Comprehensive Exam in Algebra (500)

May, 2013.

Each question is worth 20 points.

1. (a) Let $n \geq 3$. Show that the alternating group A_n is generated by 3-cycles.

   (b) Let $n \geq 5$. Let $H \leq S_n$ be a subgroup, and let $H_1 \trianglelefteq H$ be a normal subgroup such that H/H_1 is abelian. If H contains all 3-cycles, then show that H_1 also contains all 3-cycles.

   (c) Deduce that S_n is not solvable for $n \geq 5$. Also show that the commutator subgroup of S_n is A_n.

2. (a) Let $f\colon G \to G'$ be an epimorphism of groups. Let H be a Sylow p-subgroup of G. Then $f(H)$ is either the trivial group or a Sylow p-subgroup of G'.

   (b) Let G_2 be a finite group and let p a prime dividing $|G|$, the order of G. Suppose $H \trianglelefteq G$ is a normal subgroup such that p does not divide $[G : H]$. Show that all Sylow p-subgroups of G are contained in H.

3. (a) Deduce from the structure theorems for modules over a PID the following: Given any finite dimensional vector space $E \neq 0$ over the field k and $A \in \mathrm{End}_k(E)$, there exists a direct sum decomposition

$$E = E_1 \oplus \cdots \oplus E_r,$$

   where each E_i is a principal $k[A]$-submodule with invariant $q_i \neq 0$ such that $q_1 | q_2 | \cdots | q_r$. The sequence $(q_1, \ldots, q_r)$ is uniquely determined by E and A, and q_r is the minimal polynomial of A. (Note: The *invariant* of a principal $k[A]$-module M is the monic polynomial $q(t)$ of minimal degree such that $q(A)\,M = 0$.)

   (b) Let k' be an extension field of k and A be an $n \times n$ matrix with entries in k. Show that the invariants of A over k are the same as its invariants over k'.

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4. Prove that $f(x) = x^p - x - 1$ is irreducible in $\mathbb{Z}[x]$. (Hint: use Problem #5.)

5. Let k be a field of charateristic $p > 0$, and let $a \in k$. Show that the polynomial $f(x) = x^p - x - a$ either (i) splits into linear factors over k, or (ii) is irreducible over k.

6. Let k be a field of some characteristic p (which could be 0) and let n be a positive integer; in the case that $p > 0$, assume also that n is prime to p. Let ζ be a primitive nth root of unity in $\bar{k}$, the algebraic closure of k.

   (a) Show that $k(\zeta)$ is a normal extension of k.

   (b) Let $G = \mathrm{Aut}_k(k(\zeta))$. Prove that G can be identified as a subgroup of $(\mathbb{Z}/n\mathbb{Z})^\times$ (the multiplicative group of units in $\mathbb{Z}/n\mathbb{Z}$). Deduce that G is abelian.

   (c) Let $k = \mathbb{Q}$ the field of rational numbers. Assume in this case $G = (\mathbb{Z}/n\mathbb{Z})^\times$ and deduce that $\mathbb{Q}(\zeta_5) \cap \mathbb{Q}(\zeta_8) = \mathbb{Q}$. (Hint: may use $\phi(mn) = \phi(m)\phi(n)$ when m, n relatively prime.)