COMPREHENSIVE EXAM, MATHEMATICS 500
FRIDAY, MAY 22, 2009

- Justify your answers.
- In multipart problems you may assume parts you could not prove when doing other parts.
- Good luck.

Group Problem.

a. [10 points] Let H be a subgroup of a group G. Then G acts on the coset set $G/H = \{gH | g \in G\}$ by left translations. Consider this action as a map $\alpha : G \longrightarrow S(G/H)$, where $S(X)$ is the group of permutations of a set X, and prove that the kernel of α is contained in H.

b. [10 points] Let G be a finite group and H a subgroup of G such that $[G : H] = p$, where p is the smallest prime dividing $|G|$. Prove that H is normal in G.

c. [10 points] Find all groups of order p^2, where p is a prime number.

d. [10 points] Find all groups of order $2009 = 49 \cdot 41$.

Ring Problem.

a. [10 points] Let R be a ring and $R[[x]]$ be the ring of formal power series over R. Show that $\sum_{i=0}^{\infty} a_i x^i$ is a unit in $R[[x]]$ iff a_0 is a unit in R.

b. [10 points] State the Eisenstein criterion.

c. [10 points] For any prime p, show that $h(x) = \sum_{i=0}^{p-1} x^i \in \mathbb{Z}[x]$ is irreducible. Hint: write $h(x)$ as $h(x) = (x^p - 1)/(x - 1)$, and consider $h(x + 1)$.

Field Problem.

Consider the field $F = \mathbb{Q}[i, \sqrt[4]{3}]$, where $i^2 = -1$.

a. [10 points]. Show that F is a Galois extension of $\mathbb{Q}$.

b. [10 points]. Find the Galois group of F over $\mathbb{Q}$.

c. [10 points]. How many subfields does F have?