COMPREHENSIVE EXAM, MATHEMATICS 500
MONDAY, MAY 12, 2008

Justify your answers. Good luck.

Problem 1. Let S_5 be the symmetric group on 5 letters, and C_5 be the cyclic group of order 5.

a. (10 points) Find the number of group homomorphisms from C_5 to S_5.

b. (10 points) Find the number of group homomorphisms from S_5 to C_5.

Problem 2. Let G be a finite group and p a prime number.

a. (10 points) Define a p-subgroup of G. Define a Sylow p-subgroup of G.

b. (10 points) Let H be a normal p-subgroup of G. Prove that H is contained in every Sylow p-subgroup of G.

Problem 3.

a. (10 points) Let R be a commutative ring with identity. Define when an ideal $I \subset R$ is maximal and when it is prime. Does maximal imply prime? Does prime imply maximal?

b. (10 points) Let k be a field and $R = k[x]$ be the ring of polynomials with k-coefficients. Define when a polynomial $f \in k[x]$ is irreducible and show that the following three statements are equivalent: (1) $f \in k[x]$ is irreducible, (2) the principal ideal $(f) \subset R$ is prime, (3) the principal ideal $(f) \subset R$ is maximal. State the theorems you use.

c. (10 points) Show that there are field extensions of every degree of $\mathbb{Q}$.

Problem 4. Consider the field $F = \mathbb{Q}[\sqrt[2]{3}, \sqrt[3]{2}]$.

a. (10 points) Find the degree $[F : \mathbb{Q}]$.

b. (10 points) Find the Galois group of F over $\mathbb{Q}$.

c. (10 points) Is F Galois over $\mathbb{Q}$? If not, find the minimal Galois extension E of $\mathbb{Q}$ containing F.