Math 500
Comprehensive Examination
May 2007

(Answer all five questions: each question is worth 20pts.)

1.

(a) Let A and B be normal subgroups of a group G such that $A \cap B = 1$. Prove that $ab = ba$ for all $a \in A,\ b \in B$.

(b) Let G be a group such that $G = S_1 \times S_2 \times \cdots \times S_k$, where the S_i are non-abelian simple groups. Prove that G has trivial center.

(c) Let G be a group as in (b). Let $N \lhd G$ and assume that $N \cap S_i = 1$ for all i. Prove that $N = 1$.

(d) Let G be as in (b) and suppose that that N is a non-trivial normal subgroup of G. Prove that N is the direct product of certain of the S_i's.

2.

(a) Let M and N be normal subgroups of a group G such that $M \cap N = 1$. If G/M and G/N are solvable, show that G is solvable.

(b) Prove that every group of order 605 is solvable.

3.

(a) Let R be a euclidean domain, i.e., an integral domain for which the division algorithm is valid. Prove that R is a principal ideal domain.

(b) Let R be an integral domain such that the polynomial ring $R[x]$ is a principal ideal domain. Prove that R must be a field.

4. Let E denote the field extension $\mathbb{Q}(2^{\frac{1}{2}}, 2^{\frac{1}{3}})$.

(a) Find $(E : \mathbb{Q})$.

(b) Prove that $E = \mathbb{Q}(2^{\frac{1}{2}} + 2^{\frac{1}{3}})$.

(c) Let S be the smallest field containing E which is normal over $\mathbb{Q}$. Find $(S : E)$ and hence $(S : \mathbb{Q})$.

5.

(a) Describe the standard method for showing that an irreducible quintic polynomial over $\mathbb{Q}$ is not solvable by radicals and apply it to the polynomial $x^5 - 4x + 2$.

(b) Let E be a Galois extension of a field F and let $G = \mathrm{Gal}(E/F)$. Assume that p^m divides $|G|$ where p is a prime. Prove that there is a subfield K of E containing F such that $(E : K) = p^m$.

(c) Show that in the situation of (b) there need not be a subfield L of E such that $(L : F) = p^m$.