Math 500 Comprehensive Exam, May, 2006

Problem 1. Let G be a group with a normal subgroup N and subgroups $K \trianglelefteq H \leq G$.

(a) (6 points) Prove that HN/KN is isomorphic with a quotient group of H/K.

(b) (6 points) Prove that $(H \cap N)/(K \cap N)$ is isomorphic with a normal subgroup of H/K.

(c) (8 points) If H/K is nontrivial, prove that at least one of HN/KN and $(H \cap N)/(K \cap N)$ must be nontrivial.

Problem 2.

(a) (9 points) Let G be a finite group with N a normal subgroup. Let P be a Sylow p-subgroup of N. Prove that $G = N_G(P)N$, where $N_G(P)$ denotes the normalizer of P in G.

(b) (11 points) Let G be a finite group of order p^2q^2 where p, q are distinct primes. Assume that

$$p \not\equiv \pm 1 (\text{mod } q) \quad \text{and} \quad q \not\equiv \pm 1 (\text{mod } p).$$

Prove that G is abelian.

Problem 3. An ideal I of a commutative ring R is called *irreducible* if $I \neq R$ and I is *not* the intersection of two strictly larger ideals of R.

(a) (5 points) If P is a prime ideal of R, show that P is irreducible.

(b) (6 points) Give an example of an irreducible ideal which is not prime.

(c) (6 points) Use Zorn's Lemma to show that if $0 \neq r \in R$, then there is an irreducible ideal I such that $r \notin I$.

(d) (3 points) Prove that every proper ideal of R is an intersection of irreducible ideals.

Problem 4. Let S denote the subring of the direct product $\mathbb{C}[x] \times \mathbb{C}[y]$ that consists of all pairs (f, g) for which $f(0) = g(0)$.

(a) (8 points) Define a function $\phi : \mathbb{C}[x, y] \to S$ by $\phi(h) = (h(x, 0), h(0, y))$. Prove that ϕ is a surjective ring homomorphism.

(b) (4 points) Prove that $\mathbb{C}[x, y]/(xy) \cong S$.

(c) (8 points) Use (b) to give a detailed description of the prime ideals of S. Justify your answer.

Problem 5. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree n. Suppose that the Galois group G of $f(x)$ is abelian.

(a) (11 points) Prove that G has order n.

(b) (9 points) Must G be cyclic? Justify your answer.