1. Let $P$ be a Sylow $p$-subgroup of a finite group $G$ and let $N_G(P) \subseteq H \subseteq G$ be a subgroup, where $N_G(P)$ denotes the normalizer of $P$. Prove that $N_G(H) = H$.

2. Must a group of order $3 \cdot 3 \cdot 3 \cdot 5$ be nilpotent? Justify your answer.

3. Let $V$ be a vector space over the field $K$. Assume that $V$ is isomorphic to the direct sum of cyclic $K[x]$-modules

$$K[x]/(x+1)^2 \oplus K[x]/(x^2-1) \oplus K[x]/(x-1)^2.$$

(a) Determine the invariant factors and elementary divisors for $V$.

(b) Give the rational canonical form for the matrix that describes multiplication by $x$ on $V$, i.e. for the linear map $V \rightarrow V$ that maps $v \mapsto xv$.

4. Let $F$ and $K$ be fields with $F \subseteq K$.

(a) State what it means for an element $x \in K$ to be algebraic over $F$.

(b) Using the definition in (a) prove that if $x \in K$ and $y \in K$ are algebraic over $F$ then both $x + y$ and $xy$ are algebraic over $F$.

5. Let $f(x) = x^4 + 6x^2 + 1 \in \mathbb{Q}[x]$.

(a) Compute, with proof, the Galois group of the polynomial $f(x)$. You may use that the polynomial has discriminant $\Delta = 2^{14}$ and cubic resolvent $g(x) = x^3 - 12x^2 + 32x$.

(b) Let $K$ be the splitting field over $\mathbb{Q}$ of the polynomial $f(x)$. Use the Galois group obtained under (a) to determine the number of subfields $F \subseteq K$ with $[F : \mathbb{Q}] = 2$.

NOTE: If you were not able to solve (a) you may assume that the Galois group is $G \simeq A_4 \subseteq S_4$. 