MATH 500 — January 2018

Five problems, 20 points each. Maximum 100 points.

Justify all your answers!

1. Let P be a Sylow p–subgroup of a finite group G and let $N_G(P) \subseteq H \subseteq G$ be a subgroup, where $N_G(P)$ denotes the normalizer of P. Prove that $N_G(H) = H$.

2. Must a group of order $3 \cdot 3 \cdot 3 \cdot 5$ be nilpotent? Justify your answer.

3. Let V be a vector space over the field K. Assume that V is isomorphic to the direct sum of cyclic $K[x]$-modules

$$K[x]/(x+1)^2 \oplus K[x]/(x^2-1) \oplus K[x]/(x-1)^2.$$

   (a) Determine the invariant factors and elementary divisors for V.

   (b) Give the rational canonical form for the matrix that describes mulitplication by x on V, i.e. for the linear map $V \longrightarrow V$ that maps $v \mapsto xv$.

4. Let F and K be fields with $F \subset K$.

   (a) State what it means for an element $x \in K$ to be algebraic over F.

   (b) Using the definition in (a) prove that if $x \in K$ and $y \in K$ are algebraic over F then both $x + y$ and xy are algebraic over F.

5. Let $f(x) = x^4 + 6x^2 + 1 \in \mathbb{Q}[x]$.

   (a) Compute, with proof, the Galois group of the polynomial $f(x)$. You may use that the polynomial has discriminant $\Delta = 2^{14}$ and cubic resolvent $g(x) = x^3 - 12x^2 + 32x$.

   (b) Let K be the splitting field over $\mathbb{Q}$ of the polynomial $f(x)$. Use the Galois group obtained under (a) to determine the number of subfields $F \subset K$ with $[F : \mathbb{Q}] = 2$.

   NOTE: If you were not able to solve (a) you may assume that the Galois group is $G \simeq A_4 \subset S_4$.