**Math 500 Comp. Exam, January 2019**

<u>All answers must contain proper justifications.</u>

1. Let G be a p-group. Let H be a normal subgroup of G of order p. Show that H is contained in the center of G. (20 pts.)

2. Find all abelian groups, up to isomorphism, of order 360 by listing in each case the elementary divisors and the corresponding invariant factors. (20 pts.)

3. a) Show that $\mathbb{Z}[\sqrt{2}] = \{a + b\ \sqrt{2} | a, b, \in \mathbb{Z}\}$ is a Euclidean domain. (5 pts.)

   b) Consider the ring $R = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} | a, b, \in \mathbb{Z}\}$. Show that the ideal $I = (3, 2 + \sqrt{-5})$ is not principal. (10 pts.)

   c) Is it possible for $R$, as defined in b), to be a Euclidean domain with respect to some norm? Justify your answer. (5 pts.)

4. a) Find the cyclotomic polynomial $\Phi_{20}(x)$ for 20th roots of unity over any field $K$ whose characteristic is relatively prime to 20. (5 pts.)

   b) Let $F = \mathbb{Z}/p\mathbb{Z}, p$ a prime, and let $K$ be an extension of $F$ such that $[K : F] = n$. Prove that the elements of $K$ are the roots of $x^{p^n} - x = 0$. (8 pts.)

   c) Show that every irreducible factor of $\Phi_k(x), k = p^n - 1$, in $F[x]$ has degree $n$. (7 pts.)

5. Consider $f(x) = x^5 - 4x - 2 \in \mathbb{Q}[x]$.

   a) Show that $f(x)$ is irreducible in $\mathbb{Q}[x]$. (5 pts.)

   b) Let $K$ be the splitting field of $f(x)$ in $\overline{\mathbb{Q}}$. Find the Galois group $G(K/\mathbb{Q})$ of $f(x)$ over $\mathbb{Q}$. Give justifications for your answer in detail. (15 pts.)