Math 500 Comp. Exam, January 2019

All answers must contain proper justifications.

1. Let $G$ be a $p$-group. Let $H$ be a normal subgroup of $G$ of order $p$. Show that $H$ is contained in the center of $G$. (20 pts.)

2. Find all abelian groups, up to isomorphism, of order 360 by listing in each case the elementary divisors and the corresponding invariant factors. (20 pts.)

3. a) Show that $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b, \in \mathbb{Z}\}$ is a Euclidean domain. (5 pts.)

b) Consider the ring $R = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b, \in \mathbb{Z}\}$. Show that the ideal $I = (3, 2 + \sqrt{-5})$ is not principal. (10 pts.)

c) Is it possible for $R$, as defined in b), to be a Euclidean domain with respect to some norm? Justify your answer. (5 pts.)

4. a) Find the cyclotomic polynomial $\Phi_{20}(x)$ for 20th roots of unity over any field $K$ whose characteristic is relatively prime to 20. (5 pts.)

b) Let $F = \mathbb{Z}/p\mathbb{Z}$, $p$ a prime, and let $K$ be an extension of $F$ such that $[K : F] = n$. Prove that the elements of $K$ are the roots of $x^n - x = 0$. (8 pts.)

c) Show that every irreducible factor of $\Phi_k(x)$, $k = p^n - 1$, in $F[x]$ has degree $n$. (7 pts.)

5. Consider $f(x) = x^5 - 4x - 2 \in \mathbb{Q}[x]$.

a) Show that $f(x)$ is irreducible in $\mathbb{Q}[x]$. (5 pts.)

b) Let $K$ be the splitting field of $f(x)$ in $\overline{\mathbb{Q}}$. Find the Galois group $G(K/\mathbb{Q})$ of $f(x)$ over $\mathbb{Q}$. Give justifications for your answer in detail. (15 pts.)