MATH 500 — JANUARY 2017

Five problems, 20 points each. Maximum 100 points.

Justify all your answers!

1. (a) Suppose G is a group with an abelian normal subgroup N and abelian quotient group G/N. Is G an abelian group?

   (b) Suppose G has order p^3 for a prime p. Is G a nilpotent group?

2. (a) Prove that every finite field has order a power of a prime p.

   (b) Give an example of a commutative ring R of prime power order p^n which is not a field.

3. Consider the action of the group $G = GL_2(\mathbb{C})$ (2×2 invertible matrices with complex coefficients) on the set $M_2(\mathbb{C})$ (*all* 2×2 matrices with complex coefficients) defined by $g \cdot M = gMg^{-1}$. For the matrix

$$M_{ab} = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}$$

   describe (in terms of $a, b \in \mathbb{C}$) its stabilizer subgroup

$$G_{M_{ab}} = \{ g \in GL_2(\mathbb{C}) \mid gM_{ab}g^{-1} = M_{ab} \}.$$

4. Let D be a unique factorization domain. If $D' \subset D$ is a subring containing 1 is it true that D' is itself a unique factorization domain?

5. Let $p(x) \in \mathbb{Q}[x]$ be a polynomial of degree 5, with splitting field E and Galois group isomorphic to the alternating group A_5.

   (a) Is $p(x)$ irreducible over $\mathbb{Q}$?

   (b) Find $[E : \mathbb{Q}]$.

   (c) How many subfields of E have degree 12 over $\mathbb{Q}$?

   (d) Which subfields of E are normal over $\mathbb{Q}$?