MATH 500 — JANUARY 2016

Five problems, 20 points each. Maximum 100 points.

1. Prove that a group of order $25 \cdot 7 \cdot 17$ must be solvable.

2. Let $n \geq 3$ and let T denote the set of 2-element subsets of $\{1, 2, \ldots, n\}$. For $\sigma \in A_n$ (the alternating group) and $\{i, j\} \in T$, let $\sigma(\{i, j\}) = \{\sigma(i), \sigma(j)\}$.

   (a) Show that this defines an action of A_n on T.

   (b) Is this action of A_n on T transitive? Justify your answer.

3. Make $\mathbb{C}^3$ into a $\mathbb{C}[x]$-module by $f(x)\mathbf{v} = f(M)\mathbf{v}$ where $\mathbf{v} \in \mathbb{C}^3$ and

$$M = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 7 \end{pmatrix}.$$

   Find polynomials $p_i(x)$ and exponents e_i such that $\mathbb{C}^3 \cong \bigoplus_i \mathbb{C}[x]/(p_i^{e_i})$ as $\mathbb{C}[x]$-modules. Justify your answer.

4. Compute, with proof, the Galois group of $g(x) = x^3 - 3x + 1 \in \mathbb{Q}[x]$.

5. Let k be a field of characteristic $p > 0$ and let $f = x^p - x + a \in k[x]$. Suppose that $f(x)$ has no roots in k. Prove that then f is irreducible over k.