MATH 500 — JANUARY 2015

Four problems, 25 points each. Maximum 100 points.

1. Let G be a finite group with center $C(G)$. Show that G is nilpotent if and only if there exists a subgroup $A \subset C(G)$ such that G/A is nilpotent.

2. Let $p \leq q$ be odd primes. Show that a group of order $2pq$ is solvable.

3. (a) Show that $\mathbb{Z}[\sqrt{10}]$ is not a unique factorization domain.
(b) Is the polynomial $p(x) = x^3 - 4ix^2 + 16x - (1 + 3i)$ irreducible in $\mathbb{Z}[i][x]$?

4. Show that an irreducible quartic polynomial $f(x) \in \mathbb{Q}[x]$ with exactly two real roots has Galois group $G \simeq S_4$ or $G \simeq D_8$.