MATH 500 Comprehensive Exam. January 2014.

In what follows, justify the answers.

(1) Let X be a regular hexagon, that is, a polygon with 6 equal sides and equal angles.
   (a) Describe the set of *rigid* symmetries of X. ("Flipping", that is, orientation-reversing symmetries of X, *are* allowed. Hexagon is assumed to be rigid, so we do not allow symmetries that "twist or bend" the hexagon.)
   (b) Denote $G = Sym(X)$ the set of rigid symmetries of X. Prove that $Sym(X)$ is a group. What is the order of this group? Describe the action of G on X.
   (c) Let g be the rotation by $180°$. What is the centralizer of g in G?
   (d) Explicitly describe one 2-Sylow subgroup of G.
   (e) How many 3-Sylow subgroups does G have? Prove that each of them is normal in G.
   (f) List all the Sylow subgroups of G that are normal in G.
   (g) Is G solvable?
   (h) Is G nilpotent?
   (i) Suppose that each side of X can be colored in one color. In how many ways can the sides of the hexagon be colored if one is allowed to use 5 colors arbitrarily? (Possibly with repetitions.) Let $C(X)$ be the set of all possible such colorings of X. Describe the action of G on $C(X)$ induced by the action of G on X.
   (j) What is the maximal number of colorings of X (using 5 colors) such that, for any two of them, one cannot be obtained from the other by acting by any element of G?

(2) (a) For a unique factorization domain R prove that $p \in R$ is irreducible if and only if (p) is a prime ideal.
   (b) Show that the ring $\mathbb{Z}[\sqrt{-5}]$ is not a UFD.

(3) (a) Let E/F be a Galois extension of degree p^k. Prove that there exists an intermediate field K with $[E : K] = p$ and K/F Galois of degree p^{k-1}.
   (b) Determine the splitting field extension $E/\mathbb{Q}$ of the polynomial $x^4 - 2 \in \mathbb{Q}[x]$.
   (c) Give an intermediate field K with $[E : K] = 2$ and $K/\mathbb{Q}$ Galois.