MATH 500 Comprehensive Exam. January 2013.

(1) Let G be a finite group of order $|G|$, and let $Z(G)$ denote the center of G. Prove the following.
  (a) If $G/Z(G)$ is cyclic, then G is abelian.
  (b) If $|G| = pq$, where p and q are primes, then either $Z(G) = \{1\}$ or G is abelian.

(2) Show that a group of order $20 \cdot 23^r$, where r is a positive integer, is solvable.

(3) Let R be a commutative ring and let M be an R-module. M is called *projective* if there exists an R-module N such that the R-module $M \oplus N$ is free.
  Let $\mathbb{Z}$ be the ring of integers, and $\mathbb{Q}$ be the field of rational numbers viewed as a $\mathbb{Z}$-module. Is $\mathbb{Q}$ a projective $\mathbb{Z}$-module? Justify your answer.

(4) (a) Let k be a field and let U be a finite multiplicative subgroup of k. Prove that U is cyclic.
  (b) Let k^* denote the set of units in k. Assume that k is a finite field. Show that k^* is a cyclic group.

(5) (a) Let k be a field and let $f(x) \in k[x]$ be such that its derivative $f'(x)$ is not the null polynomial. Prove that the following are equivalent.
    (i) $f(x)$ has a multiple root in the algebraic closure of k.
    (ii) $f(x)$ and $f'(x)$ have a common root in the algebraic closure of k.
    (iii) The greatest common divisor of $f(x)$ and $f'(x)$ in $k[x]$ is of degree ≥ 1.
  (b) A polynomial over k is called *separable* if its roots in the algebraic closure of k are distinct. Prove the following statements.
    (i) An irreducible polynomial over a field k of characteristic 0 is separable.
    (ii) Let k be a field of characteristic $p > 0$ and $f(x)$ be an irreducible polynomial in $k[x]$. Suppose that $f(x)$ cannot be expressed as a polynomial in x^p with coefficients in k. Then $f(x)$ is separable.
  (c) Let p be a prime number. Show that the polymonial $f(x) = x^{p-1} + \ldots + x + 1$ is irreducible over $\mathbb{Q}$.

(6) Consider the polynomial $f(x) = x^4 - 2$ over $\mathbb{Q}$.
  (a) Find the splitting field K of $f(x)$ and its degree over $\mathbb{Q}$.
  (b) Let G be the Galois group of the field extension $\mathbb{Q} \subseteq K$. Find the generators and relators for G. Is it isomorphic to the dihedral group D_8? Justify your answer.