MATH 500 — JANUARY 2012

Five problems, 20 points each. Maximum 100 points.

1. (a) Show that every group of order 77 is Abelian.

(b) Show that every group of order 135 is nilpotent.

2. Let G be a finite group, N a normal subgroup of G, p a prime number, and P a p-Sylow subgroup of N. Recall that

$$N_G(P) := \{g \in G : \; gPg^{-1} = P\}$$

is the normalizer of P in G.

(a) Define an action of G on the set of p-Sylow subgroups of N.

(b) Show that $G = N_G(P)N$.

3. Let R be a PID. Using just the definition of "PID", prove:

(a) R has no infinite strictly ascending chain of ideals

$$I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_n \subsetneq \cdots .$$

(b) Every nonzero prime ideal I of R is a maximal ideal of R.

(c) Give an example of a nonzero prime ideal of $\mathbb{Z}[x]$ that is not a maximal ideal of $\mathbb{Z}[x]$.

4. Let R and S be commutative rings with $1 \neq 0$ and let $f : R \longrightarrow S$ be a ring homomorphism.

(a) Prove that if Q is a prime ideal of S, then its preimage $f^{-1}(Q)$ is a prime ideal of R.

(b) Give an example where the preimage of a maximal ideal of S is not a maximal ideal of R.

(c) Give an example where the image $f(P)$ of a prime ideal P of R is not a prime ideal of S.

5. Consider the field $\mathbb{F}_2$ of two elements.

(a) Show that the polynomial $x^3 + x + 1 \in \mathbb{F}_2[x]$ is irreducible.

(b) Show that $x^3 + x + 1$ has a root $\alpha \in \mathbb{F}_8$ and that $\mathbb{F}_8$ is a splitting field of $x^3 + x + 1$ over $\mathbb{F}_2$.

(c) Find the matrix of the Frobenius automorphism of $\mathbb{F}_8$ with respect to the basis $1, \alpha, \alpha^2$ (and $\mathbb{F}_8$ as vector space over $\mathbb{F}_2$).