Comprehensive Exam, MATH 500, January 20, 2011.

1. (a) For a group G, let L_G denote the unordered list of all groups that occur as quotient groups of G. We require that no two groups in the list be isomorphic to each other. Show that for any two groups G_1 and G_2 of order 51, the two lists are the same, i.e. $L_{G_1} = L_{G_2}$. For any group G of order 51, write this list explicitly. Justify the answer.

   (b) Give explicit examples of two groups G_1 and G_2 of order 39 whose unordered lists L_{G_1} and L_{G_2} are not the same.

2. Recall that a series of subgroups of a group G is called normal if each term of the series is normal in G. A series is called subnormal if each term of the series is normal in the next (bigger) term. For each $n \geq 1$, answer the following question. Is it true that each subnormal series of the symmetric group S_n ending with the trivial group is a normal series? (Hint: treat the cases $n = 1, 2, 3, 4$ and $n \geq 5$ separately)

3. (a) A commutative ring R is called Noetherian if each ideal in R is finitely generated. If R is a Noetherian commutative ring, show that every ascending chain $I_1 \subseteq I_2 \subseteq \ldots$ of ideals in R stabilizes, i.e. there exists an index i such that $I_i = I_{i+1} = \ldots$.

   (b) Suppose R is a Noetherian integral domain. Prove that each nonzero nonunit element $r \in R$ has a factorization into irreducibles, i.e there exist irreducible elements $q_1, \ldots, q_n \in R$ such that $r = q_1 \ldots q_n$.

4. Let p be a prime number and q be a power of p. Denote $\bar{\mathbb{F}}_q$ the algebraic closure of the field $\mathbb{F}_q$ with q elements. Let $\varphi(x) \in \mathbb{F}_q[x]$ be a polynomial of the form
   $\varphi(x) = a_0 x + a_1 x^p + \ldots + a_r x^{p^r}$, $r \geq 1$, $a_r \neq 0$.

   (a) Show that φ is additive: $\varphi(u+v) = \varphi(u) + \varphi(v)$ for $u, v \in \bar{\mathbb{F}}_q$.

   (b) Show that the roots of φ in $\bar{\mathbb{F}}_q$ have multiplicity one if and only if $a_0 \neq 0$.

   (c) Prove that the set of roots of φ in $\bar{\mathbb{F}}_q$ is an additive subgroup of $\bar{\mathbb{F}}_q$.