COMPREHENSIVE EXAM, MATH 500, JANUARY 20, 2010.

1. (a) Let P be a Sylow p-subgroup of a finite group G and let $N_G(P) \leq H \leq G$ where $N_G(P)$ denotes the normalizer. Prove that $N_G(H) = H$.
   (b) Show that there are no simple groups of order 616.
2. Let G be a group with a composition series of finite length, $l(G)$, and let $N \lhd G$.
   (a) Show that N has a composition series.
   (b) Show that G/N has a composition series.
   (c) Prove that $l(G) = l(N) + l(G/N)$.
3. Let E be the field extension $\mathbb{Q}(\sqrt{3} - \sqrt{2})$, where $\mathbb{Q}$ is the rational field.
   (a) Prove that $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.
   (b) Find $(E : \mathbb{Q})$.
   (c) Find the irreducible polynomial of $\sqrt{3} - \sqrt{2}$.
   (d) If K is a field such that $\mathbb{Q} \leq K \leq E$, prove that K is normal over $\mathbb{Q}$.
4. (a) Prove that for any commutative ring R and a proper ideal I in R, there exists a maximal ideal in R containing I.
   (b) Prove that if J is a maximal ideal in a commutative ring R, then the quotient R/J is a field.