COMPREHENSIVE EXAM, MATHEMATICS 500
WEDNESDAY, JANUARY 14, 2009

Justify your answers. Good luck.

Problem 1 [10 points]. Find the number of symmetries of a cube (*i.e.* the number of permutations of the 8 vertices which take edges to edges and faces to faces).

Problem 2. Let H and K be normal subgroups of a group G. Assume that $HK = G$ and $H \cap K = \{1\}$.
a [10 points]. Prove that $hk = kh$ for any $h \in H$ and $k \in K$.
b [10 points]. Prove that G is isomorphic to $H \times K$.

Problem 3 [10 points]. Let P be a Sylow p-subgroup of a finite group G and N be a normal subgroup of G. Prove that $P \cap N$ is a Sylow p-subgroup of N.
Hint: you might want to consider the subgroup PN/N of G/N.

Problem 4.
a [10 points]. Define an Euclidean domain and a PID. Prove that an Euclidean domain is a PID.
b [10 points]. Prove that $\mathbb{Q}[x]$ is an Euclidean domain and hence a PID.
c [10 points]. Find a generator of the ideal $\langle x^3 - 3x + 2, x^4 - 1, x^6 - 1\rangle$ of $\mathbb{Q}[x]$.

Problem 5. Consider $F = \mathbb{Q}[\sqrt[4]{5}]$ as an extension of $\mathbb{Q}$.
a [10 points]. Find the Galois group of F over $\mathbb{Q}$.
b [10 points]. Find the normal closure E of F.
c [10 points]. Find the Galois group of E over $\mathbb{Q}$.