COMPREHENSIVE EXAM, MATHEMATICS 500
SATURDAY JANUARY 26TH, 2008

[redacted]

Problem 1.
a. (10 points) Is there a simple group of order 2008?
b. (10 points) Is there a non-abelian group of order 2008?
Justify. Note: the prime factorization of 2008 is $2008 = 2^3 \cdot 251$.

Problem 2.
a. (10 points) Let $\mathbb{F}_5$ be the field with 5 elements and $\mathrm{GL}(2, \mathbb{F}_5)$ the group of invertible 2×2 matrices with coefficients in $\mathbb{F}_5$. Find the order of $\mathrm{GL}(2, \mathbb{F}_5)$.
b. (10 points) Find the number of matrices conjugate to $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ in $\mathrm{GL}(2, \mathbb{F}_5)$.

Problem 3.
a. (10 points) Factor $p(x) = x^{11} - 1$ into a product of irreducible polynomials in $\mathbb{Q}[x]$. Show that your factors are, in fact, irreducible.
b. (10 points) Now consider $p(x) = x^{11} - 1$ as an element of $\mathbb{R}[x]$. How many maximal ideals does $\mathbb{R}[x]/(x^{11} - 1)$ have?

Problem 4. Consider the ring $E = \mathbb{Q}[\sqrt[3]{2}] = \mathbb{Q}[x]/(x^3 - 2)$.
a. (10 points) Show that E is a field.
b. (10 points) Is E Galois over $\mathbb{Q}$?
c. (10 points) Let F be the smallest Galois extension of $\mathbb{Q}$ containing E (so $F = E$ if the answer to b. is positive). Find the degree $[F : \mathbb{Q}]$.
d. (10 points) How many subfields does F have?