Comprehensive Examination
Math 500 – January 2007

(Answer all five questions; each one is worth 20 pts)

1. Let n be a positive integer and p an odd prime such that $p \leq n$.

(a) Prove that every element of order p in the symmetric group S_n is an even permutation.

(b) Prove that the alternating group A_n can be generated by p-cycles.

(c) Show that the number of p-cycles in S_n is given by

$$l(n,p) = \binom{n}{p} \cdot (p-1)!.$$

(d) Prove that the number of elements of order p in S_n is

$$\sum_{i=1}^{r} \frac{l(n,p) \cdot l(n-p,p) \cdot \quad \cdots \quad \cdot l(n-(i-1)p,p)}{i!},$$

where r is the greatest integer less than or equal to $\frac{n}{p}$.

2.

(a) Let A and B be solvable subgroups of a group G and suppose that $A \triangleleft G$. Prove that AB is solvable.

(b) Let G be a group of order 2007. Prove that G has subgroups P and Q with orders 9 and 223 respectively, such that $Q \triangleleft G$ and $G = PQ$.

(c) Prove that any group of order 2007 is solvable with derived length at most 2. Then use semidirect products to give an example of a *non-abelian* group of order 2007.

3. Let $I_1, I_2, \ldots$ be ideals in an integral domain.

(a) If $I_1 \cap I_2 \cap \cdots \cap I_m = 0$, show that at least one I_i must equal 0.

(b) Give an example to show that the conclusion of 3(a) may be false for an intersection of *infinitely* many ideals.

(c) Assume that $I_1 \cap I_2 \cap \cdots = 0$, with infinitely many non-zero I_i's. Then prove that $I_k \cap I_{k+1} \cap \cdots = 0$ for all positive integers k.

4. Let E denote the field $\mathbb{Q}(2^{1/3}, 3^{1/2})$.

(a) Find $(E : \mathbb{Q})$.

(b) Show that $E = \mathbb{Q}(c)$ where $c = 3^{1/2} - 2^{1/3}$.

(c) Find the irreducible polynomial of c over $\mathbb{Q}$.

5.

(a) Let E be a Galois extension of a field F with characteristic 0. Prove that there is a unique smallest subfield K such that $F \subseteq K \subseteq E$, K is normal over F and E is subradical over K. [You will need the result in 2(a)].

(b) Let f be an irreducible polynomial over $\mathbb{Q}$ which has degree 5 and at least two complex roots. Prove that $\mathrm{Gal}(f)$ has order $10, 20, 60$ or 120.