Math 500

Comprehensive Exam

January 2006

1. Let G be a group of order $p^n > 1$ where p is a prime.

(a) Prove that the center of G has order > 1. [6 pts]

(b) Prove that there is a series of *normal* subgroups of G

$$1 = G_0 < G_1 < \cdots < G_n = G$$

such that $|G_{i+1} : G_i| = p$. [7 pts]

(c) If H is a subgroup of G, show that it is *subnormal* in G, i.e., there is a chain of subgroups $H = H_0 \lhd H_1 \lhd \cdots \lhd H_m = G$, where $m \leq n$. [7 pts]

2. (a) Let P be a Sylow p-subgroup of a finite group G and let $N \lhd G$. Prove that $P \cap N$ is a Sylow p-subgroup of N. [7 pts]

(b) Show that the conclusion of 2(a) may be false if the subgroup N is not normal in G. [4 pts]

(c) Prove that every group of order 666 is solvable. [9 pts]

3. Prove or disprove each of the following statements:

(a) $\mathbb{Z}[x, y, z]/(x - y)$ is an integral domain; [5 pts]

(b) every maximal ideal of $\mathbb{R}[x]$ has the form $(x - a)$ where $a \in \mathbb{R}$; [3 pts]

(c) $\mathbb{C}[x, y]$ is a principal ideal domain; [5 pts]

(d) $\mathbb{Q}[x, y]/(xy - 1)$ is a unique factorization domain. [7 pts]

4. (a) Show from first principles that a finite field has order equal to a power of a prime. [6 pts]

In the rest of the problem E denotes the field $\mathbb{Q}(\sqrt{5}-\sqrt{3})$.

(b) Prove that $\sqrt{3} \in E$ and $\sqrt{5} \in E$. [3 pts]

(c) Find $(E:\mathbb{Q})$. [4 pts]

(d) Find the irreducible polynomial of $\sqrt{5}-\sqrt{3}$ over $\mathbb{Q}$. [7 pts]

5. Let f be a polynomial of degree 5 over $\mathbb{Q}$ and assume that the Galois group of f is isomorphic with the alternating group A_5. Let K be the splitting field of f.

(a) Prove that f is irreducible over $\mathbb{Q}$. [5 pts]

(b) Show that no subfield of K can have degree 2, 3 or 4 over $\mathbb{Q}$. [5 pts]

(c) Determine the number of subfields of K which have degree 12 over $\mathbb{Q}$. [7 pts]

(d) How many normal subfields does K have? [3 pts]