MATH 500 — AUGUST 2017

Five problems, 20 points each. Maximum 100 points.

Justify all your answers!

1. (a) Let $u_n = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$ denote the upper triangular nilpotent matrix with 1s just above the diagonal and 0s elsewhere. Show that if c is any nonzero complex number and I_n is the $n \times n$ identity matrix, then $(cI_n + u_n)^k \neq I_n$ for all $k > 0$.

   (b) Let $GL_n(\mathbb{C})$ denote the group of invertible $n \times n$ matrices with complex coefficients (with matrix multiplication as the group operation). Prove that for every $k > 0$, if $\Phi : \mathbb{Z}/k\mathbb{Z} \to GL_n(\mathbb{C})$ is any group homomorphism, there exists some $g \in GL_n(\mathbb{C})$ such that $g\Phi(m)g^{-1}$ is a diagonal matrix for all $m \in \mathbb{Z}/k\mathbb{Z}$.

   (c) Prove by example that the conclusion of part (b) can fail if $GL_n(\mathbb{C})$ is replaced by $GL_n(F)$ for appropriate choices of integers k and n and finite field F.

2. Must a group of order $3 \cdot 5 \cdot 7$ be solvable? Justify your answer.

3. Let $A = \begin{pmatrix} -2 & -2 & -1 \\ 0 & -4 & -1 \\ 0 & +4 & 0 \end{pmatrix}$. Make $V = \mathbb{C}^3$ into a $\mathbb{C}[x]$-module by

$$f(x)v := f(A) \cdot v \quad \text{(matrix multiplication) for } f(x) \in \mathbb{C}[x], v \in V.$$

   Find an elementary divisor decomposition of the module V. Justify your answer.

4. Consider the ring $R = \mathbb{C}[x, y, z]/\langle z^2 - xy \rangle$. Show that R is not a UFD.

5. Let $K = \mathbb{Q}(\omega)$ where $\omega = e^{2\pi i/17}$.

   (a) Prove that K contains a unique subfield L such that $[L : \mathbb{Q}] = 8$.

   (b) Is L Galois over $\mathbb{Q}$? Justify.