MATH 500 — August 2018

Five problems, 20 points each. Maximum 100 points.

Justify all your answers!

1. Let G be a finite group of order p^2q^2, with $p \neq q$ prime numbers. Show that there is a Sylow subgroup of G which is normal in G.

2. Let R be a ring with identity 1. A element $x \in R$ is called *nilpotent* if $x^n = 0$, for some positive integer n. Denote by $N \subset R$ the set of nilpotent elements. Show that:

   (a) If x is nilpotent then $1 - x$ is a unit;

   (b) If R is commutative, then $N \subset R$ is an ideal;

   (c) If R is commutative, then R/N has exactly one nilpotent element.

3. Let V denote the vector space over $\mathbb{R}$ of real polynomials of degree $\leq n$. Let $T : V \to V$ be the linear map given by

$$T(p(x)) = p'(x).$$

   (a) Find the Jordan canonical form of T;

   (b) Find the rational canonical form of T.

4. Let L be a Galois extension of $\mathbb{Q}$ of order 100. Show that there exists a chain of extensions:

$$\mathbb{Q} = K_0 \subsetneq K_1 \subsetneq K_2 \subsetneq K_3 \subsetneq K_4 = L$$

   where each K_{i+1} is a Galois extension of K_i.

5. Show that the polynomial $p(x) = x^6 - 3 \in \mathbb{Q}[x]$ is irreducible and determine its Galois group.