1. Let $G$ be a finite group of order $p^2q^2$, with $p \neq q$ prime numbers. Show that there is a Sylow subgroup of $G$ which is normal in $G$.

2. Let $R$ be a ring with identity 1. A element $x \in R$ is called nilpotent if $x^n = 0$, for some positive integer $n$. Denote by $N \subset R$ the set of nilpotent elements. Show that:

(a) If $x$ is nilpotent then $1 - x$ is a unit;

(b) If $R$ is commutative, then $N \subset R$ is an ideal;

(c) If $R$ is commutative, then $R/N$ has exactly one nilpotent element.

3. Let $V$ denote the vector space over $\mathbb{R}$ of real polynomials of degree $\leq n$. Let $T : V \rightarrow V$ be the linear map given by

$$T(p(x)) = p'(x).$$

(a) Find the Jordan canonical form of $T$;

(b) Find the rational canonical form of $T$.

4. Let $L$ be a Galois extension of $\mathbb{Q}$ of order 100. Show that there exists a chain of extensions:

$$\mathbb{Q} = K_0 \subsetneq K_1 \subsetneq K_2 \subsetneq K_3 \subsetneq K_4 = L$$

where each $K_{i+1}$ is a Galois extension of $K_i$.

5. Show that the polynomial $p(x) = x^6 - 3 \in \mathbb{Q}[x]$ is irreducible and determine its Galois group.