MATH 500 — AUGUST 2016

Five problems, 20 points each. Maximum 100 points.

1. A *short exact sequence* of groups is given by group homomorphisms:

$$K \xrightarrow{i} G \xrightarrow{j} H \qquad (\dagger)$$

where i is injective, j is surjective and $\ker j = \text{im } i$. The short exact sequence is called *split* if there exists a group homomorphism $s : H \to G$ such $j \circ s = \text{id}$.

   (a) Show that there is a split short exact sequence:

$$A_n \xrightarrow{i} S_n \xrightarrow{j} \mathbb{Z}_2 .$$

   (b) Show that if the short exact sequence ($\dagger$) is split then G is isomorphic to a semi-direct product $G \simeq H \ltimes K$.

2. Let D be a PID and let R be an integral domain containing D as subring. Show that if d is a g.c.d in D of elements $a, b \in D$ then d is also a g.c.d of a and b in R.

3. Determine the structure of the abelian group $\mathbb{Z}^3/K$ where K is the subgroup generated by $(2, 1, -3)$ and $(1, -1, 2)$.

4. (a) Let $u = e^{2\pi i/12}$, a primitive 12th root of unity. Show that $[\mathbb{Q}(u), \mathbb{Q}] = 4$ and determine the minimal polynomial of u over $\mathbb{Q}$.

   (b) Let F be a subfield of E and assume that $E = F(u)$ where u is algebraic of odd degree. Show that $E = F(u^2)$.

5. Compute, with proof, the Galois group of $g(x) = x^3 - 4x + 1 \in \mathbb{Q}[x]$.