MATH 500 — AUGUST 2015

Four problems, 25 points each. Maximum 100 points.

1. Let G be a group of order $5 \cdot 13 \cdot 43 \cdot 73$. Determine the number of elements of order five.

2. Let G be a finite group and N a normal subgroup of G. Prove or disprove:
(a) G is nilpotent if and only if both N and G/N are nilpotent.
(b) G is solvable if and only if both N and G/N are solvable.

3. Let R be an integral domain. Let $f \in R[x]$ be a nonzero polynomial such that there exist a nonzero $d \in R$ and polynomials $g, h \in R[x]$ of degree less than f such that $df = gh$.
(a) Show that if R is a unique factorization domain then f is the product of two polynomials in $R[x]$ of degree less than f.
(b) Use part (a) with $f = x^2 - 5$ to show that $\mathbb{Z}[\sqrt{20}]$ is not a unique factorization domain.

4. Let G be a finite group. Show that there exist fields L and K such that L is an extension of K with Galois group G.