MATH 500 — AUGUST 2014

Four problems, 25 points each. Maximum 100 points.

1. Let G be the unique simple group of order 168. Determine the number of elements of order seven.

2. Let G be a finite group, let P be a Sylow p-subgroup of G and let $H = N_G(P)$ be the normalizer of P in G. Show that, for all $g \in G$, $gHg^{-1} = H$ if and only if $g \in H$.

3. (a) Prove that in a unique factorization domain R, an element x is irreducible if and only if it is prime.
(b) Give an example of a commutative ring R and an element $x \in R$ such that x is irreducible but not prime.

4. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible quartic polynomial with splitting field extension $L/\mathbb{Q}$.
(a) List the possible degrees for the extension $L/\mathbb{Q}$ and for each degree the possible Galois groups of the extension.
(b) Let $f(x)$ have the property that $\mathbb{Q}(\alpha) \cap \mathbb{Q}(\beta) = \mathbb{Q}$, for any two distinct roots α and β of $f(x)$. Determine the Galois group of $L/\mathbb{Q}$.