Comprehensive Exam in Algebra (500)

August, 2013.

Answer all six questions. Each question is worth 20 points.

1. Let G be a finite group acting on a finite set X. Let
$$X^G = \{ x \in X \mid gx = x \text{ for all } g \in G \},$$
and for any $x \in X$ let
$$G \cdot x = \{ gx \mid g \in G \}, \qquad \text{and} \qquad G_x = \{ g \in G \mid gx = x \}.$$
   (a) Prove that
$$\text{(i)} \quad X = X^G \cup \Big(\bigcup_{\substack{x \in X \\ \text{Card}\, G \cdot x \neq 1}} G \cdot x \Big), \qquad \text{(ii)} \quad \text{Card}(G \cdot x) = [G : G_x].$$
   (Card denotes "cardinality".)

   (b) Suppose that G is a p-group. Prove that $\text{Card}\, X \equiv \text{Card}\, X^G \mod p$.

   (c) Prove that the center of a non-trivial p-group is non-trivial. (Hint: take $X = G$ with a suitable action and use (b).)

   (d) Deduce that any p-group is nilpotent.

2. (a) Let G be a finite group, and let $H \trianglelefteq G$ be a normal subgroup. Suppose that p is a prime dividing the order of G, but p does not divide $[G : H]$. Show that all Sylow-p-subgroups of G are contained in H.

   (b) Suppose G has order $p^2 q$, where p and q are distinct primes. Show that G is not simple.

3. Determine a complete list of all non-isomorphic abelian groups of order 1800.

4. Let R be an integral domain with quotient field F. Let $f(x) \in R[x]$ be a monic polynomial, and assume $f(x) = g(x)h(x)$ where g, h are monic polynomials in $F[x]$ of smaller degree than $f(x)$. Prove that if $g(x) \notin R[x]$, then R is not a UFD. Deduce that $\mathbb{Z}[2\sqrt{2}]$ is not a UFD.

5. (a) Let k be a field, and let $f(x) \in k[x]$ of degree n. Let L denote the splitting field of $f(x)$ over k and let G be the Galois group of L/k. Show how to identify G with a subgroup of S_n.

   (b) With hypotheses as in (a), let $f(x) = x^3 + ax + b$, and assume that $f(x)$ is irreducible in $k[x]$. Discuss the possibilities for G and its corresponding orders, and how these depend on a and b. If α is a root of $f(x)$, is $k(\alpha)$ normal over k?

   (c) Determine the Galois group of $x^3 - x + 1$ over $\mathbb{Q}$.

6. Let F be a field, and let G be a finite group of automorphisms of F of order n. Let $k = F^G$ be the fixed field of G.

   (a) Let $\alpha \in F$ and let $\sigma_1, \dots, \sigma_r$ be a maximal set of elements of G such that $\sigma_1\alpha, \dots, \sigma_r\alpha$ are distinct. Show that every $\tau \in G$ induces a bijection on $\{\sigma_1\alpha, \dots, \sigma_r\alpha\}$ via multiplication on the left.

   (b) Prove that every $\alpha \in F$ is the root of a polynomial $f(x) \in k[x]$, where all roots are distinct and contained in F, and $\deg f(x) \leq n$. (Hint: use (a).)

   (c) Deduce that F is a finite Galois extension of k of degree n with Galois group G.