Comprehensive Exam in Algebra (500)

August, 2012.

Each question is worth 25 points.

1. (a) Let H, K be subgroups of a group G. Show that if $H \trianglelefteq G$, then HK is also a subgroup of G.

   (b) Given an example to show that HK can fail to be a subgroup if neither H or K are normal.

   (c) Prove thatif $H \trianglelefteq G$ of prime index p, then for any subgroup $K \leq G$ either (i) $K \leq H$ or (ii) $G = HK$ and $|K : K \cap H| = p$.

2. (a) Let P be a p-Sylow subgroup of a finite group G. Show that if $N \trianglelefteq G$ is a normal subgroup of G, then $P \cap N$ is a p-Sylow subgroup of N.

   (b) Give an example to show that (a) can fail if N is not normal.

   (c) Show that every group of order $460 = 4 \cdot 5 \cdot 23$ is solvable.

3. Let R be an integral domain.

   (a) Given an element $x \in R$ define what it means for x to be irreducible, and what it means for x to be prime. By proof or counterexmaple, determine whether irreducible implies prime, and whether prime implies irreducible.

   (b) Show that if R is a PID then $x \in R$ is prime if and only if it is irreducible.

   (c) Let A denote the ring $\mathbb{Z}(\sqrt{-5}) = \{\, a + b\sqrt{-5} \mid a, b \in \mathbb{Z} \,\}$. Prove that A is not a principal ideal domain.

4. Let $E = \mathbb{Q}(a)$ where $a = \sqrt{1 + \sqrt{2}}$.

   (a) Find the irreducible polynomial of a.

   (b) What is $(E : \mathbb{Q})$.

   (c) Identify the Galois group of $E/\mathbb{Q}$.

   (d) How many subfields of E are there?