MATH 500 COMPREHENSIVE EXAM

AUGUST 2011

Problem 0.1. Let G be a finite group and let S be a finite set upon which G operates.

(1) State and prove the orbit formula for the operation of G on S.
(2) Prove that if G is a p-group then it has a non trivial center.
(3) Write down the definition of a nilpotent group.
(4) Prove that a finite p-group is nilpotent

Problem 0.2. (1) Classify all finite groups of order 57.

(2) Let G and N be two groups. Define the statement "G is operating on N by automorphisms." Define the semidirect product of G and N.
(3) Let k be a field and let T denote the group of nonsingular 2×2 matrices over k. Let D be the subgroup of diagonal matrices in T and let U be the subgroup of T whose diagonal entries are all 1. Show that D operates on U and that T is the semidirect product of these two groups.

Problem 0.3. Let k be a field and let R be the polynomial domain in four variables $k[x, y, z, w]$.

(1) Define a homomorphism ϕ from R to $k(x, y, z)$ by the equations $\phi(x) = x, \phi(y) = y, \phi(z) = z, \phi(w) = xy/z$. Describe the kernel I of ϕ. Show that it is a principal ideal and give its generator.
(2) Prove that the ideal I is prime and show that R/I is not a unique factorization domain.

Problem 0.4. (1) State Eisenstein's criterion for a monic integral polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and prove it.

(2) Let $f = x^5 + 5x^4 + 10x^3 + 10x^2 + x - 5 \in \mathbb{Q}[x]$. Prove that f is irreducible over $\mathbb{Q}$.
(3) Prove that the Galois group of the splitting field of f, the polynomial of (2) above, is S_5 the symmetric group on 5 letters.

Problem 0.5. (1) Give a precise statement of Zorn's lemma.

(2) Let R be a commutative ring with unit. A subset S of R is said to be multiplicatively closed if $s \in S$ and $t \in S$ imply that $st \in S$. Let S be a multiplicatively closed subset of R not

containing 0. Prove that an ideal J maximal with respect to the property that $S \cap J$ is empty is prime.

(3) Let R be commutative with unit and let S be a multiplicatively closed subset of R which does not contain 0. Use Zorn's lemma to show that there is a prime of R that does not meet S.

(4) Show that the intersection of an arbitrary descending chain of prime ideals is prime. Use Zorn's lemma to prove that any commutative ring contains a minimal prime ideal.