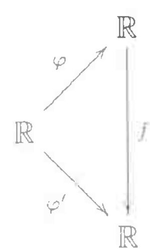
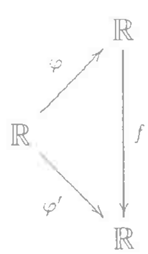
Comprehensive Exam, MATH 500, August 23, 2010.

1. Let p be an odd prime and S_p the symmetric group of p objects.
   (a) If P is a Sylow p-subgroup of S_p, what is the order of P?
   (b) How many cycles of length p are there in S_p?
   (c) Calculate the number n_p of Sylow p-subgroups in S_p.
   (d) Use this to prove *Wilson's Theorem*: $(p-1)! \equiv -1 \mod p$.
2. Let $\mathbb{Q}$ be the field of rational numbers, and $R = \mathbb{Q}[x,y]/\langle x^2, y^3\rangle$.
   (a) Show that for any prime ideal $\mathfrak{p}$ we have $x, y \in \mathfrak{p}$.
   (b) Show that R is a local ring: R has a unique maximal ideal.
3. Let p be a prime, $\mathbb{F}_p$ the prime field of order p, and $\mathbb{E} = \mathbb{F}_q$, for $q = p^n, n \geq 1$.
   (a) Prove that $\mathbb{E}^\times$, the multipliciative group of non zero elements of $\mathbb{E}$, is a cyclic group. (You can use the classification of finite Abelian groups here.)
   (b) Show that $\mathbb{E}$ is the splitting field of the polynomial $x^q - x \in \mathbb{F}_p[x]$.
4. (a) Define $obj(\mathcal{C})$ to be the set of all continuous maps $\varphi : \mathbb{R} \to \mathbb{R}$. For $\varphi, \varphi' \in obj(\mathcal{C})$, let $Hom(\varphi, \varphi')$ be the set of all commutative diagrams



such that f is continuous, and define the composition of



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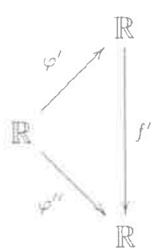
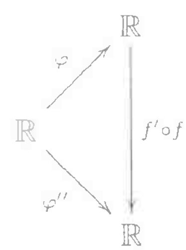


to be



Does this define a category? Justify the answer.

   (b) Define $obj(\mathcal{C})$ to be $\mathbb{Q}$. For $a, b \in \mathbb{Q}$, define

$$Hom(a,b) := \begin{cases} \text{the set consisting of one pair } (a,b) & \text{if } a < b \\ \emptyset & \text{if } a \geq b \end{cases}$$

and the composition $(a,b) \circ (b,c)$ to be (a,c). Does this define a category? Justify the answer.

   (c) For the ones above that are categories, determine whether those categories have initial objects, terminal (or final) objects, zero objects.