COMPREHENSIVE EXAM, MATHEMATICS 500
WEDNESDAY, AUGUST 19, 2009
SOLUTIONS

Group Problem.
a. The generator of of C_5 should go to an element of order 5 or order 1 in S_5. Answer: $4! + 1 = 25$.
b. The kernel of a homomorphism is a normal subgroup of S_5, hence either S_5 or A_5, but $|S_5/A_5| = 2 \not| 5$. Hence the only homomorphism is the trivial one.
c. $|\{(123)^k(45)^l\}| = 6$
d. The number of Sylow subgroups of order 7 is 8 and each contains 6 elements of order 7. Answer: 48.

Ring Problem.
a. $\mathbb{R}[x]$ is a PID. $\mathbb{R} \times \mathbb{R}$ is not a domain.
b. $(x + a)$ or $(x^2 + 2bx + c)$, where $b^2 < c$.
c. $\mathbb{R} \times 0$ and $0 \times \mathbb{R}$.

Field Problem. First of all, $\sqrt{2}, \sqrt{3} \in \mathbb{Q}[\sqrt{2} + \sqrt{3}]$.
Hence $\mathbb{Q}[\sqrt{2} + \sqrt{3}] = \mathbb{Q}[\sqrt{2}, \sqrt{3}]$.
a. The degree is 4; the Galois group is $C_2 \times C_2$.
b. Since $\mathbb{Q}$ has no subfields, subfields of F correspond to subgroups of its Galois group over $\mathbb{Q}$. Answer: $\mathbb{Q}$, $\mathbb{Q}[\sqrt{2}]$, $\mathbb{Q}[\sqrt{3}]$, $\mathbb{Q}[\sqrt{6}]$, $\mathbb{Q}[\sqrt{2} + \sqrt{3}]$.
c. $p(x) = \prod_{\pm\pm}(x \pm \sqrt{2} \pm \sqrt{3}) = x^4 - 10x^2 + 1$.