COMPREHENSIVE EXAM, MATHEMATICS 500
WEDNESDAY, AUG 20, 2008

Justify your answers. Good luck.

Problem 1. Let p be a prime number.

a. (10 points) Define a Sylow p-subgroup of a finite group.

b. (10 points) Let $\mathbb{F}_p$ be a finite field with p elements, $GL(2, \mathbb{F}_p)$ the group of invertible 2x2 matrices with $\mathbb{F}_p$-coefficients, and $U(2, \mathbb{F}_p) = \{ \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) | \, a \in \mathbb{F}_p \}$ the subgroup of upper-triangular unipotent matrices. Show that $U(2, \mathbb{F}_p)$ is a Sylow p-subgroup of $GL(2, \mathbb{F}_p)$.

c. (10 points) Find the number of Sylow p-subgroups of $GL(2, \mathbb{F}_p)$.

Problem 2. (10 points) Find (a) all homomorphisms of additive groups $\mathbb{Z} \to \mathbb{Q}$ and (b) all homomorphisms of rings $\mathbb{Z} \to \mathbb{Q}$.

[In this problem ring homomorphisms are not required to map 1 to 1]

Problem 3. Let R be an integral domain.

a. (10 points) Define when an element $x \in R$ is irreducible and when it is prime. Prove or give a counter-example: irreducible $\Rightarrow$ prime, prime $\Rightarrow$ irreducible.

b. (10 points) Define when R is Euclidean and show that if R is Euclidean then it is a principal ideal domain.

c. (10 points) Show that if R is a principal ideal domain and $x \in R$ then x is prime $\Leftrightarrow$ x is irreducible.

Problem 4. Let p be a prime number and consider a polynomial

$$f_p(x) = x^4 + p^2 \in \mathbb{Q}[x]$$

a. (10 points) Find the splitting field E of f_p.

b. (10 points) Find the Galois group of E over $\mathbb{Q}$.

c. (10 points) Is f_p irreducible in $\mathbb{Q}[x]$?

[The answers might depend on p]