Math 500
Comprehensive Examination
August 2007

(Answer all five questions: each question is worth 20pts.)

1. Let G be a finite group.

(a) Let P be a Sylow p-subgroup of G and write $N = N_G(P)$ for its normalizer. If H is a subgroup such that $N \leq H \leq G$, prove that $H = N_G(H)$.

(b) Assume that every subgroup S of G is subnormal, i.e., there is a series in G containing S. Prove that G is a direct product of p-groups for various primes p.

2.

(a) State Eisenstein's Criterion for irreducibility of a polynomial $f \in R[x]$, where R is a unique factorization domain, and then prove its validity.

(b) Let $f = x^5 + 5x^4 + 10x^3 + 10x^2 + x - 5 \in \mathbb{Q}[x]$. Prove that f is irreducible over $\mathbb{Q}$. [You may assume Gauss's Lemma].

(c) Prove that $\mathrm{Gal}(f) \simeq S_5$ where f is the polynomial in 2(b).

3.

(a) Let R denote the ring $\mathbb{Z}(\sqrt{-5}) = \{a + b\sqrt{-5} | a, b \in \mathbb{Z}\}$. Prove that R is *not* a unique factorization domain.

(b) Suppose that R is a commutative ring with identity such that the polynomial ring $R[x]$ is a unique factorization domain. Prove that R is a unique factorization domain.

4.

(a) Give a careful statement of Zorn's Lemma.

(b) Let R be a commutative ring with identity and let $\{P_\lambda \mid : \lambda \in \Lambda\}$ be a chain of prime ideals of R. Prove that $\bigcap_{\lambda \in \Lambda} P_\lambda$ is a prime ideal of R.

(c) Prove that every commutative ring R with identity has a minimal prime ideal by using Zorn's Lemma.

5. Let E be a Galois extension of a field F.

(a) If $(E : F) = pq$ where p, q are primes and $p > q$, show that there is a subfield K such that $(K : F) = q$ and $(E : K) = p$.

(b) Assume that $(E : F) = p^k > 1$ where p is a prime. Prove that there is a subfield S of E which is normal over F and satisfies $(E : S) = p$.