Math 500
Comprehensive Examination
August 2006

(Answer all five questions: each question is worth 20pts.)

1.

(a) Let G be a finite group and let H be a proper subgroup of G. Prove that G cannot equal the union of all the conjugates of H.

(b) Suppose that G is a finite group with even order. Prove that the number of conjugacy classes in G with odd order is odd.

2.

(a) Let G be a group with order $p^m q$ where p and q are primes and $p > q$. Prove that $G = PQ$ where P and Q are subgroups of orders p^m and q respectively and P is normal in G.

(b) Suppose that $m = 2$ in 2(a), so that $|G| = p^2 q$, and assume that $p \not\equiv \pm 1 (mod\ q)$. Prove that G is abelian.

3. Let $R = M_2(\mathbb{C})$ be the ring of all 2×2 matrices over the complex field $\mathbb{C}$. Put $X = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \in R$ and define a function $\theta : \mathbb{C}[x] \to R$ by the rule $\theta(f) = f(X)$.

(a) Prove that θ is a ring homomorphism.

(b) Identify the kernel of θ.

(c) Describe the prime ideals of $\mathrm{Im}(\theta)$.

4. Let $a = \sqrt{1 + \sqrt{2}}$ and put $E = \mathbb{Q}(a)$.

(a) Find the irreducible polynomial of a.

(b) Find $(E : \mathbb{Q})$.

(c) Identify the Galois group of E over $\mathbb{Q}$.

5.

(a) Describe the standard method for showing that an irreducible quintic polynomial is not solvable by radicals and apply it to the polynomial $x^5 - 4x + 2$.

(b) Determine whether $x^5 + 5x^3 - x^2 - 5 \in \mathbb{Q}[x]$ is solvable by radicals. Justify your answer.