Logic Comprehensive Exam (Math 570)
August 19, 2011

Do all five problems. Explain your answers. The problems have equal weight.

Notation: L is a first order language with equality, $\mathbb{N}$ is the set $\{0, 1, 2, \dots\}$ of natural numbers.

1. Let L have just a constant symbol 0, a binary function symbol $+$, and a binary relation symbol R, and consider all L-structures $\mathcal{A} = (\mathbb{N}; 0, +, X)$ where 0 and $+$ have their usual meaning, and $X \subseteq \mathbb{N}^2$ is the interpretation of R.

   (a) Indicate an L-formula $\phi(x, y)$ that defines in every $\mathcal{A}$ the set $\{(m, n) \in \mathbb{N}^2 : \ m \leq n\}$.

   (b) Indicate an L-sentence σ such that for every $\mathcal{A}$,

$$\mathcal{A} \models \sigma \iff X \text{ is infinite.}$$

2. Suppose L has just a unary function symbol f and let $\mathcal{A} = (A, f)$ be an L-structure such that f is a permutation of A. Suppose further that there is no positive integer n such that f^n is the identity on A. (Here $f^1 = f$ and $f^{n+1} = f \circ f^n$.) Show that there is a countable L-structure $\mathcal{B} = (B, g)$ that satisfies the same L-sentences as $\mathcal{A}$ with an element $b \in B$ such that $b, g(b), g^2(b), \dots$ are all distinct.

3. Suppose L has just a unary relation symbol U and a binary relation symbol $<$. Let T be the theory whose models are the structures $\mathcal{A} = (A; P, <)$ where $(A, <)$ is a dense linear ordering without endpoints, and $P = U^{\mathcal{A}}$ is a nonempty proper subset of A such that whenever $a < b \in P$, then $a \in P$.

   (a) Find all complete L-theories extending T, by indicating for each such complete extension T' a sentence σ' such that $T \cup \{\sigma'\}$ axiomatizes T'. (You may use the $\aleph_0$-categoricity of the theory of dense linear orderings without endpoints.)

   (b) Indicate for each T' as in (a) a model of T'.

4. Let L be a finite language with at least a constant symbol 0 and a unary function symbol S, and let T be a consistent theory in L.

(a) What does it mean for a function $f : \mathbb{N} \to \mathbb{N}$ to be representable *as a function* in T?

(b) Use your definition in (a) to show that if $f, g : \mathbb{N} \to \mathbb{N}$ are representable as functions in T, then the composition $f \circ g$ is representable in T as a function.

(c) Suppose that, for all $i, j \in \mathbb{N}$ with $i \neq j$, $T \vdash S^i 0 \neq S^j 0$. Show that if $f : \mathbb{N} \to \mathbb{N}$ is representable in T as a function and T is finitely axiomatizable, then f is computable in the intuitive sense.

5. Let E be an equivalence relation on $\mathbb{N}$ which is recursively enumerable as a subset of $\mathbb{N}^2$, that is, for some recursive functions $f, g : \ \mathbb{N} \to \mathbb{N}$, the following holds for all $m, n \in \mathbb{N}$:

$$mEn \iff \text{there is } k \text{ such that } f(k) = m, \ g(k) = n.$$

Suppose E has only finitely many classes. Show that $E \subseteq \mathbb{N}^2$ is recursive.