Do all five problems. Explain your answers. The problems have equal weight.

Notation: $L$ is a first order language with equality, $\mathbb{N}$ is the set $\{0, 1, 2, \ldots\}$ of natural numbers.

1. Let $L$ have just a constant symbol 0, a binary function symbol $+$, and a binary relation symbol $R$, and consider all $L$-structures $A = (\mathbb{N}; 0, +, X)$ where 0 and $+$ have their usual meaning, and $X \subseteq \mathbb{N}^2$ is the interpretation of $R$.

   (a) Indicate an $L$-formula $\phi(x, y)$ that defines in every $A$ the set $\{(m, n) \in \mathbb{N}^2 : m \leq n\}$.

   (b) Indicate an $L$-sentence $\sigma$ such that for every $A$, $A \models \sigma \iff X$ is infinite.

2. Suppose $L$ has just a unary function symbol $f$ and let $A = (A, f)$ be an $L$-structure such that $f$ is a permutation of $A$. Suppose further that there is no positive integer $n$ such that $f^n$ is the identity on $A$. (Here $f^1 = f$ and $f^{n+1} = f \circ f^n$.) Show that there is a countable $L$-structure $B = (B, g)$ that satisfies the same $L$-sentences as $A$ with an element $b \in B$ such that $b, g(b), g^2(b), \ldots$ are all distinct.

3. Suppose $L$ has just a unary relation symbol $U$ and a binary relation symbol $<$. Let $T$ be the theory whose models are the structures $A = (A; P, <)$ where $(A, <)$ is a dense linear ordering without endpoints, and $P = U^A$ is a nonempty proper subset of $A$ such that whenever $a < b \in P$, then $a \in P$.

   (a) Find all complete $L$-theories extending $T$, by indicating for each such complete extension $T'$ a sentence $\sigma'$ such that $T \cup \{\sigma'\}$ axiomatizes $T'$. (You may use the $\aleph_0$-categoricity of the theory of dense linear orderings without endpoints.)

   (b) Indicate for each $T'$ as in (a) a model of $T'$.

4. Let $L$ be a finite language with at least a constant symbol 0 and a unary function symbol $S$, and let $T$ be a consistent theory in $L$.

   (a) What does it mean for a function $f : \mathbb{N} \rightarrow \mathbb{N}$ to be representable as a function in $T$?

   (b) Use your definition in (a) to show that if $f, g : \mathbb{N} \rightarrow \mathbb{N}$ are representable as functions in $T$, then the composition $f \circ g$ is representable in $T$ as a function.

   (c) Suppose that, for all $i, j \in \mathbb{N}$ with $i \neq j$, $T \vdash S^i 0 \neq S^j 0$. Show that if $f : \mathbb{N} \rightarrow \mathbb{N}$ is representable in $T$ as a function and $T$ is finitely axiomatizable, then $f$ is computable in the intuitive sense.

5. Let $E$ be an equivalence relation on $\mathbb{N}$ which is recursively enumerable as a subset of $\mathbb{N}^2$, that is, for some recursive functions $f, g : \mathbb{N} \rightarrow \mathbb{N}$, the following holds for all $m, n \in \mathbb{N}$:

   $$m En \iff \text{there is } k \text{ such that } f(k) = m, \ g(k) = n.$$  

Suppose $E$ has only finitely many classes. Show that $E \subseteq \mathbb{N}^2$ is recursive.