Logic Comprehensive Exam (Math 570),
August 2007

There are 5 problems. Each problem is worth 20 points, for a total of 100 points. To receive credit, each of your solutions must be justified.

Convention

In the exercises, L will be a language and $=$ is considered a logical symbol. Any model-theoretic structure is by convention non-empty.

Another word for 'recursive' is 'computable'.

1 Exercise

Two sets $A, B \subseteq \mathbb{N}$ are said to be *recursively isomorphic* if there is a recursive bijection $h\colon \mathbb{N} \to \mathbb{N}$ such that $h[A] = B$.

(a) Show that if A and B are infinite recursive subsets of $\mathbb{N}$ with infinite complements $\mathbb{N} \setminus A$ and $\mathbb{N} \setminus B$, then A and B are recursively isomorphic.

(b) Describe all recursive isomorphism classes of subsets of $\mathbb{N}$.

2 Exercise

Let L be a language with only finitely many non-logical symbols and let T be a decidable L-theory. Show that there is a complete decidable L-theory $T' \supseteq T$.

3 Exercise

Let L be the language whose non-logical symbols are a binary predicate symbol $<$ and a unary predicate symbol P. Let $\mathcal{Q}$ be L-structure $\mathcal{Q} = (\mathbb{Q}, <^{\mathcal{Q}}, P^{\mathcal{Q}})$, where $<^{\mathcal{Q}}$ denotes the usual strict ordering on $\mathbb{Q}$ and $P^{\mathcal{Q}} = \{q \in \mathbb{Q} \mid q < 0\}$.

(a) Is there an L-formula $\phi(x)$ defining the set $\{1\}$ in $\mathcal{Q}$?

(b) Is there an L-formula $\psi(x)$ defining the set $\{0\}$ in $\mathcal{Q}$?

(c) Indicate a finite set Σ of L-sentences such that for all L-sentences σ we have $\Sigma \vdash \sigma \iff \mathcal{Q} \models \sigma$.

4 Exercise

Let L be the language whose only non-logical symbol is a binary relation symbol $<$ and let σ be an L-sentence. Suppose that for all n there is a model $\mathcal{M} = (M, <^{\mathcal{M}})$ of σ such that $<^{\mathcal{M}}$ linearly orders M and $|M| \geq n$. Show that there is a model $\mathcal{M} = (M, <^{\mathcal{M}})$ of σ, linearly ordered by $<^{\mathcal{M}}$, with distinct elements $a_0, a_1, a_2, \ldots$ such that

$$\ldots <^{\mathcal{M}} a_2 <^{\mathcal{M}} a_1 <^{\mathcal{M}} a_0.$$

5 Exercise

Let Σ be a finite consistent set of sentences in a language $\mathcal{L}$ with only finitely many non-logical symbols, including a constant symbol 0 and a unary function symbol S.

(a) Define what it means for a set $A \subseteq \mathbb{N}$ to be *representable* in Σ.

(b) If $A \subseteq \mathbb{N}$ and $B \subseteq \mathbb{N}$ are representable in Σ, does it follow that the difference set $A \setminus B$ is representable in Σ?