LOGIC COMPREHENSIVE EXAM (MATH 570), AUGUST 2006

There are 5 problems. Each problem is worth 20 points, for a total of 100 points. To receive credit, each of your solutions must be justified.

Convention. In the exercises L will be a language and $=$ is considered a logical symbol. Any model-theoretic structure is by convention non-empty.

1. EXERCISE

Let for each $n \in \mathbb{N}$ a function $f_n : \mathbb{N} \to \mathbb{N}$ be given. Show that there is a function $g : \mathbb{N} \to \mathbb{N}$ such that for each n and all $k \geq n$, $f_n(k) < g(k)$.

2. EXERCISE

Let L be the first order language whose non-logical symbols consist of a constant symbol 0 for the number zero, a unary function symbol S for the successor function, a binary predicate symbol $<$ for the ordering relation, and two binary function symbols $+$ and $\times$ for addition and multiplication respectively. Let Σ be any set of sentences in L such that $\Sigma \vdash \sigma$ whenever σ is a quantifier free sentence in L that is true in the standard model of arithmetic $(\mathbb{N}, 0, S, <, +, \times)$. Let also Σ contain the sentence

$$\forall x\, \forall y\, \forall z\, (x + y = x + z \to y = z).$$

(1) Let $f : \mathbb{N}^k \to \mathbb{N}$ be a function. Define what it means for f to be *representable (as a function) in* Σ.
(2) Define $f : \mathbb{N}^2 \to \mathbb{N}$ by $f(n, m) = \max\{0, n - m\}$. Show that f is representable in Σ.

3. EXERCISE

Let $\mathcal{L}$ be a language with at least one constant symbol. Let $\phi(x)$ be a quantifier free formula. Show that

$$\vdash \exists x\, \phi$$

if and only if there exist variable free terms $t_1, \dots, t_n$ such that

$$\vdash \phi(t_1) \vee \cdots \vee \phi(t_n).$$

4. EXERCISE

Let $\mathcal{L}$ be the language consisting of one unary function symbol f. Let $\mathcal{M} = (\mathbb{R}^2, f^{\mathcal{M}})$ be an $\mathcal{L}$-structure with $f^{\mathcal{M}}(x,y) = (x,0)$.

(i) Find a decidable set Σ of $\mathcal{L}$-sentences that hold in $\mathcal{M}$ and such that any two countable models of Σ are isomorphic. (You can use Church's thesis to prove decidability of Σ.)

(ii) Show that for a set of $\mathcal{L}$-sentences Σ as in (i) we have

$$\Sigma \vdash \sigma \text{ if and only if } \mathcal{M} \models \sigma.$$

5. EXERCISE

Let $\mathcal{L}$ be a language consisting of a binary relation symbol $<$ and a unary function symbol f. Let $\mathcal{M}$ be an $\mathcal{L}$-structure whose underlying set is the set of integers $\mathbb{Z}$. Let $<^{\mathcal{M}}$ be the usual ordering on $\mathbb{Z}$, and let $f^{\mathcal{M}}(m) = m + 2$. Is the set of even integers definable in this structure?