LOGIC COMPREHENSIVE EXAM (MATH 570), MAY 2008

There are 5 problems. Each problem is worth 20 points, for a total of 100 points. To receive credit, each of your solutions must be justified.

Convention. In the exercises, L will be a language and $=$ is considered a logical symbol. Any model-theoretic structure is by convention non-empty.

Also, 'total recursive functions' are partial recursive functions that happen to be defined everywhere. Another standard terminology for recursive is 'computable'.

1. Exercise

Let $A \subseteq \mathbb{N}^2$ be recursively enumerable and suppose that for every n the section $A_n = \{k \in \mathbb{N} \mid (n,k) \in A\}$ is infinite. Show that there exists an infinite recursively enumerable set $B \subseteq \mathbb{N}$ such that $B \neq A_n$ for all n.

2. Exercise

Let L be a language and let $L' = L \cup \{<, f\}$, where $<$ is a new binary relation symbol and f a new unary function symbol.

Let T be the L-theory whose axioms are

$$\exists x_1 \ldots \exists x_n \bigwedge_{i \neq j} x_i \neq x_j$$

for all $n = 1, 2, \ldots$.

Let also S be the L'-theory whose axioms are

$$\forall x\, \forall y\, (x < y \;\vee\; y < x \;\vee\; x = y),$$

$$\forall x\, \forall y\, \neg(x < y \;\wedge\; y < x),$$

$$\forall x\, \forall y\, \forall z\, (x < y \;\wedge\; y < z \rightarrow x < z),$$

$$\forall x\, (x < fx).$$

Suppose that σ is an L sentence such that $S \vdash_{L'} \sigma$. Show that also $T \vdash_L \sigma$.

3. Exercise

Suppose that L has just a unary function symbol f, and let $\mathcal{A} = (A, f)$ be an L-structure such that f is a permutation of A. Suppose further that there is no positive integer n such that f^n is the identity on A. (Here $f^1 = f$ and $f^{n+1} = f \circ f^n$.) Show that there is a countable L-structure $\mathcal{B} = (B, g)$ that satisfies the same L-sentences as $\mathcal{A}$ and an element $b \in B$ such that $b, g(b), g^2(b), \dots$ are all distinct.

4. Exercise

Let $(\mathbb{Q}, <)$ be the set of all rational numbers equipped with the usual strict linear ordering.

(i) (7 points) Find all sets $A \subseteq \mathbb{Q}$ that are definable (without parameters) in the structure $(\mathbb{Q}, <)$.

(ii) (13 points) Find all binary relations $R \subseteq \mathbb{Q}^2$ that are definable (without parameters) in the structure $(\mathbb{Q}, <)$.

5. Exercise

Let $L = \{E\}$, where E is a binary predicate symbol. Let T be the theory in the language L whose models are the structures (A, E) with E an equivalence relation on A such that for every integer $n \geq 1$, there is exactly one equivalence class of cardinality n.

(i) Indicate an axiomatization of T.

(ii) How many countable models does T have, up to isomorphism?