LOGIC COMPREHENSIVE EXAM (MATH 570)
MAY 2007

There are 5 problems. Each problem is worth 20 points, for a total of 100 points. To receive credit, each of your solutions must be justified.

Convention. In the exercises, L will be a language and $=$ is considered a logical symbol. Any model-theoretic structure is by convention non-empty.

Also, 'total recursive functions' are partial recursive functions that happen to be defined everywhere. Another standard terminology for recursive is 'computable'.

1. Exercise

Suppose that E is an equivalence relation on $\mathbb{N}$ which is recursively enumerable as a subset of $\mathbb{N}^2$, i.e., for some total recursive functions $f, g\colon \mathbb{N} \to \mathbb{N}$ we have

$$nEm \iff \exists k \, \big(f(k) = n \ \& \ g(k) = m\big).$$

Assume moreover that E has only finitely many classes. Show that E is recursive as a subset of $\mathbb{N}^2$.

2. Exercise

Let L be the language whose non-logical symbols are: a constant symbol 0, a unary function symbol S, a binary predicate symbol $<$, and two binary function symbols $+$ and $\cdot$. Let Σ be a set of sentences in L such that Σ proves any universal sentence in L that is true in the standard model of arithmetic $(\mathbb{N}, 0, S, <, +, \cdot)$, with the usual interpretation of the non-logical symbols of L.

(a) Let A be a subset of $\mathbb{N}^k$ where $k \geq 1$ is an integer. Define what it means for A to be representable in Σ.

(b) Suppose $B \subseteq \mathbb{N}$ is representable in Σ and $A = \{n^2 : n \in B\}$. Show that A is representable in Σ.

3. Exercise

Suppose T is a theory in a countable language and that all models of T of cardinality $\aleph_1$ are pairwise elementarily equivalent. Show that, for all infinite models $\mathcal{A}$ and $\mathcal{B}$ of T, $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent.

4. Exercise

Suppose Σ is a set of L-sentences with the following properties:
(a) for any $\sigma, \tau \in \Sigma$, either $\sigma \to \tau$ or $\tau \to \sigma$ is logically valid;
(b) for each $\sigma \in \Sigma$, σ has a model.
Show that Σ has a model.

5. Exercise

Suppose L is a language and T a consistent L-theory with only finitely many logically inequivalent complete extensions $T' \supseteq T$.

Show that there are sentences $\sigma_1, \ldots, \sigma_n$ such that every complete extension $T' \supseteq T$ is equivalent to one of $T \cup \{\sigma_i\}$.