LOGIC COMPREHENSIVE EXAM (MATH 570), MAY 2006

There are 5 problems. Each problem is worth 20 points, for a total of 100 points. To receive credit, each of your solutions must be justified.

In the exercises L will be a language and $=$ is considered a logical symbol. Any model-theoretic structure is by convention non-empty.

Let $\mathcal{M}$ be an L-structure. Recall that a subset A of the underlying set M of $\mathcal{M}$ is **definable in** $\mathcal{M}$ if there exists an L-formula $\phi(x)$ such that for any $a \in M$ we have

$$a \in A \;\Leftrightarrow\; \mathcal{M} \vDash \phi(a).$$

1. Exercise

Assume $f : \mathbb{N} \to \mathbb{N}$ is a total computable function, $g : \mathbb{N} \to \mathbb{N}$ is an injective total computable function, $g[\mathbb{N}]$ is computable, and $g(n) \leq f(n)$ for all $n \in \mathbb{N}$.

Show that also $f[\mathbb{N}]$ is computable.

2. Exercise

For an L-sentence σ we let $S(\sigma)$ be the set of all positive integers n such that σ has a model of cardinality n. Find σ in some language L such that $S(\sigma) = \{2n \mid n \geq 1\}$.

3. Exercise

Let L contain a binary relation symbol $<$ and let σ be an L-sentence such that for every model $\mathcal{M}$ of σ, the interpretation of $<$ in $\mathcal{M}$ is a well-ordering of the underlying set M of $\mathcal{M}$. Show that there is a natural number K such that any model of σ has cardinality less than K.

4. Exercise

Let L contain the constant symbol 0 and the unary function symbol S.

(i) Let Σ be a set of L-sentences. State what it means for a relation $R \subseteq \mathbb{N}^k$ to be representable in Σ.

(ii) Assume that L contains additionally binary function symbols $+$, $\cdot$ and a binary relation symbol $<$. Let $\mathcal{N} = \langle \mathbb{N}, 0^{\mathcal{N}}, S^{\mathcal{N}}, +^{\mathcal{N}}, \cdot^{\mathcal{N}}, <^{\mathcal{N}} \rangle$ be the L-structure in which S is interpreted as the successor function $S^{\mathcal{N}}(x) = x + 1$ and 0, $+$, $\cdot$, $<$ have their usual interpretations. Show that there is a set $R \subseteq \mathbb{N}$ which is definable in $\mathcal{N}$, but not representable in Σ for any finite set of sentences Σ that are satisfied in $\mathcal{N}$.

5. Exercise

Let L consist of a binary predicate symbol $<$ and a constant symbol 0. Consider the structure $\mathcal{M} = (\mathbb{Q}, <^{\mathcal{M}}, 0^{\mathcal{M}})$ where $\mathbb{Q}$ is the set of all the rational numbers, $<$ is interpreted as the usual ordering of $\mathbb{Q}$, and 0 is interpreted as the rational number 0. Find all subsets of $\mathbb{Q}$ definable in $\mathcal{M}$.