LOGIC COMPREHENSIVE EXAM (MATH 570), FEBRUARY 4, 2006

There are 5 problems. Each problem is worth 20 points, for a total of 100 points. To receive credit, each of your solutions must be justified.

1. Problem

Let L be a language. For every formula $\phi(x, y_1, \ldots, y_n)$ in the language L, we denote by $\exists! x\ \phi$ the formula

$$\exists x\ (\phi(x, y_1, \ldots, y_n)\ \wedge\ \forall z\ (\phi(z, y_1, \ldots, y_n) \to x = z)).$$

Let now $\theta(x, y)$ be an L-formula with free variables among x, y. Write an L-sentence σ that is true in an L-structure $\mathcal{M}$ if and only if there is a unique pair (a, b) of elements of $\mathcal{M}$ such that $\mathcal{M} \models \theta[a, b]$.

Which of the following formulas

$$\sigma, \quad \exists! x\ \exists! y\ \theta(x, y), \qquad \exists! y\ \exists! x\ \theta(x, y)$$

are equivalent? Give either a proof or a counter example.

2. Problem

(i) Let $R \subseteq \mathbb{N}$ be infinite. Assume R is the range of a total computable function from $\mathbb{N}$ to $\mathbb{N}$. Show that there exists a total computable function $f : \mathbb{N} \to \mathbb{N}$ which is injective and whose range is R.

(ii) Let $S \subseteq \mathbb{N}$ be the range of a total computable function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) \leq f(n+1)$ for all $n \in \mathbb{N}$. Show that S is computable.

3. Problem

Let T be a theory in a language L, and let Φ, Ψ be two sets of L-formulas with free variables among $x_1, \ldots, x_n$. Suppose that for any model $\mathcal{M}$ of T and for any elements $a_1, \ldots, a_n$ of $\mathcal{M}$ we have

> $\mathcal{M} \models \phi[a_1, \ldots, a_n]$ for all formulas $\phi \in \Phi$ if and only if $\mathcal{M} \models \psi[a_1, \ldots, a_n]$ for some formula $\psi \in \Psi$.

Show that there exist finite subsets $\Phi_0 \subseteq \Phi$ and $\Psi_0 \subseteq \Psi$ such that

$$T \models \forall x_1 \cdots \forall x_n\ (\bigwedge \Phi_0 \leftrightarrow \bigvee \Psi_0).$$

4. PROBLEM

Let L be a language consisting of one binary function symbol $\cdot$. Consider the L-structure $\mathcal{N} = (\mathbb{N}, \cdot^{\mathcal{N}})$ where $\cdot^{\mathcal{N}} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is the usual multiplication. Show that the set $\{2, 3\}$ is not definable in this structure.

5. PROBLEM

Let L be a language containing the constant symbol 0 and the unary function symbol S. Let Σ be a set of L-sentences.

(i) Define what it means for a function $F : \mathbb{N}^k \to \mathbb{N}$ to be representable in Σ.

(ii) Assume that the usual addition $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is representable in Σ. Let $F_1, F_2 : \mathbb{N} \to \mathbb{N}$ be two functions representable in Σ. Show that the function $F_1 + F_2 : \mathbb{N} \to \mathbb{N}$ given by

$$(F_1 + F_2)(n) = F_1(n) + F_2(n)$$

is representable in Σ.