LOGIC COMPREHENSIVE EXAM (MATH 570), JANUARY 23, 2008

There are 5 problems. Each problem is worth 20 points, for a total of 100 points. To receive credit, each of your solutions must be justified.

Convention. In the exercises, L will be a language and $=$ is considered a logical symbol. Any model-theoretic structure is by convention non-empty.

Another word for 'recursive' is 'computable'.

1. Exercise

Let L be the language consisting of a single binary relation symbol E and let Σ be a set of L-sentences stating that E is an equivalence relation with infinitely many classes and all of whose equivalence classes are infinite.

(a) For which infinite cardinal numbers κ are all models of Σ of cardinality κ isomorphic?

(b) Is Σ complete?

(c) Is $\mathrm{Th}(\Sigma)$ decidable?

2. Exercise

Let L be the language $\{0, 1, +, \cdot, <\}$ and let $\mathcal{N}$ be the standard model with underlying set $\mathbb{N}$ and the usual interpretations of the symbols.

(a) Show that there is an L-structure $\mathcal{M} = \langle M; 0^{\mathcal{M}}, 1^{\mathcal{M}}, +^{\mathcal{M}}, \cdot^{\mathcal{M}}, <^{\mathcal{M}}\rangle$ that is elementarily equivalent to $\mathcal{N}$ and has an infinite prime, where an *infinite prime* of $\mathcal{M}$ is an element $p \in M$ such that

$$\mathcal{M} \models S^n 0 < p$$

for all $n \in \mathbb{N}$, and

$$\mathcal{M} \models \forall x \, \forall y \, (x \neq p \; \wedge \; y \neq p \rightarrow x \cdot y \neq p).$$

(b) Can $\mathcal{M}$ be taken to be countable? Of cardinality $\aleph_1$?

3. Exercise

Let L be the language $\{<, U\}$, where $<$ is a binary relation symbol and U a unary relation symbol. Indicate an L-sentence σ such that for all sets $X \subseteq \mathbb{R}$,

$$\langle \mathbb{R}, <, X\rangle \models \sigma \iff X \text{ is infinite.}$$

4. Exercise

Let L be a language with only finitely many non-logical symbols.

(a) Show that for every L-sentence σ and every natural number N, there is an L-sentence τ such that $\vdash \sigma \leftrightarrow \tau$ and such that the Gödel number of τ is greater than N.

(b) Suppose that $f\colon \mathbb{N} \to \mathbb{N}$ is a strictly increasing recursive function, i.e., $m < n \Longrightarrow f(m) < f(n)$. Show that $\mathrm{Im} f$ is a recursive subset of $\mathbb{N}$.

(c) Let T be an L-theory with a recursively enumerable axiomatization. Show that T has a recursive axiomatization.

5. Exercise

Let L be a language for number theory whose non-logical symbols include a binary relation symbol $<$, a unary function symbol S and a constant symbol 0. Let Σ be a set of L-sentences.

(a) Define what it means for a set $R \subseteq \mathbb{N}^2$ to be *representable* in Σ.

(b) Suppose that the following sentences are provable from Σ:

$$\forall x\ \neg(x < 0),$$
$$\forall x\ (x < S^n 0 \leftrightarrow \bigvee_{0 \le i < n} x = S^i 0),$$

where $n = 1, 2, \ldots$. Suppose also that the binary relation $R \subseteq \mathbb{N}^2$ is representable in Σ, and let

$$U = \{a \in \mathbb{N} \mid \exists b < a\ (a, b) \in R\}.$$

Show that U is representable in Σ. You should give an explicit formula representing U involving one that represents R and verify that it works.