LOGIC COMPREHENSIVE EXAM (MATH 570), JANUARY 2007

There are 5 problems. Each problem is worth 20 points, for a total of 100 points. To receive credit, each of your solutions must be justified.

Convention. In the exercises $\mathcal{L}$ will be a language and $=$ is considered a logical symbol. Any model-theoretic structure is by convention non-empty.

Also, 'total recursive functions' are partial recursive functions that happen to be defined everywhere. Another standard terminology for recursive is 'computable'.

1. Exercise

Prove that there exists a function $f : \mathbb{N} \to \mathbb{N} \setminus \{0\}$ such that for any total recursive function $g : \mathbb{N} \to \mathbb{N}$ we have

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0.$$

2. Exercise

Let $\mathcal{L}$ be the language consisting of one binary relation symbol $<$.

(i) Find two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ and an $\mathcal{L}$-sentence σ such that

(a) $<^{\mathcal{M}}$ and $<^{\mathcal{N}}$ are strict linear orders and
(b) $\mathcal{M}$ is a substructure of $\mathcal{N}$ and
(c) $\mathcal{M} \models \sigma$ and $\mathcal{N} \models \neg\sigma$.

(ii) Consider the $\mathcal{L}$-structure $\mathcal{R} = (\mathbb{R}, <^{\mathcal{R}})$ consisting of the reals with the usual strict linear order relation. Let $\mathcal{Q} = (\mathbb{Q}, <^{\mathcal{Q}})$ be the substructure of $\mathcal{R}$ whose underlying set consists of all the rational numbers. Show that for any $\mathcal{L}$-sentence σ

$$\mathcal{Q} \models \sigma \text{ iff } \mathcal{R} \models \sigma.$$

3. Exercise

Suppose that $\prec$ is a strict linear order of $\mathbb{N}$ which is recursively enumerable, i.e., for some total recursive functions $\phi, \psi\colon \mathbb{N} \to \mathbb{N}$ we have

$$n \prec m \iff \exists k \; \big(\phi(k) = n \;\&\; \psi(k) = m\big).$$

Show that the set $\mathbb{A} = \{(n,m) \in \mathbb{N} \times \mathbb{N} \bigm| n \prec m\}$ is recursive.

4. Exercise

Let $\mathcal{L} = \{\cdot, ^{-1}, 1\}$ be the language of groups consisting of a binary function symbol, a unary function symbol, and a constant symbol. Show that there exists no $\mathcal{L}$-formula $\phi(x,y)$ such that for every group G and $g, h \in G$,

$$G \models \phi(g,h) \iff g \text{ and } h \text{ have the same order in } G.$$

(Recall that the order of an element g of a group G is the smallest $n \in \mathbb{N}$, $n \geq 1$, with $g^n = 1$, if such an n exists, and ∞ otherwise.)

5. Exercise

Let Σ be a finite consistent set of sentences in a language $\mathcal{L}$ with only finitely many non-logical symbols, including a constant symbol 0 and a unary function symbol S.

(i) Define what it means for a relation $R \subseteq \mathbb{N}^k$ to be *representable* in Σ.

(ii) Show that each representable set $R \subseteq \mathbb{N}$ is recursive. It suffices to describe an algorithm for determining whether or not a given $n \in \mathbb{N}$ belongs to R.