Introduction
Stochastic modeling is used wherever modeling inputs are stochastically determined. In the insurance business, it is most commonly used for interest-sensitive products where incomes and benefits depend on economic scenarios. Theoretically, nested stochastic modeling is required when one stochastically determined input relies on the value of another stochastically determined parameter. For instance, the investment rate on capital may require projections of stochastic scenarios, each of which hedging strategies were developed by further stochastic modeling. As insurance industry is moving towards more detailed and complicated modeling practice, the computational burden of stochastic modeling is expected to increase exponentially.

Aim
Since the insurance industry is experiencing difficulty with nested stochastic modeling, the aim of this study is to investigate a variety of techniques and to recommend to practitioners most efficient methods based on the reduction of run time and the improvement on computational efficiency. Below is an overview of the general structure of a nested simulation.

As a part of a larger research project led by Professor Feng, we investigate the risk capital calculation for Guaranteed Lifetime Withdrawal Benefit (GLWB) as a model example for nested stochastic modeling. In a nutshell, the GLWB is an investment guarantee written on a variable annuity contract. Policyholders deposit premiums in investment accounts, which are pooled together and invested in the equity market. Without any investment guarantee, the insurer would merely charge fees and distribute agent commission to agents. The risk capital for the GLWB can be summarized into three steps:

1. Determine the cash flows from each policyholder’s account, including withdrawal, interest, and surrender charges.
2. Determine the risk-neutral values of the GLWB liability at the lowest surplus or deficiency in each scenario. The risk-neutral value (v) of the GLWB rider depends on three factors, account value (F), guarantee base (G) and time to maturity (T).
3. Take the average of discounted outcomes less incomes as an estimate of risk-neutral values of the GLWB rider.

Current market practice
The current market practice on the nested simulation of GLWB liabilities is showed below:

Process of Nested simulations

Outer loops:
- Project equity returns for each period for the next 50 years using some stochastic equity models.
- Determine the cash flows from each policyholder’s account, including withdrawals, interests, rider charges, management fees, and surrender charges.
- The change in surplus for each period is determined by the following recursive relation.

\[ \Delta_S = \text{Expenses} + \text{Investment income on cash flows} + \text{Investment income on surplus} \]

Change in surplus
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- Fee income + Surrender charge - GLWB withdrawals

Monte Carlo Method can be summarized into three steps:
1. Generate economic scenarios for further future (which may not the same as the projections in the outer loops).
2. Determine incomes and outgoes under each scenario.
3. Take the average of discounted outcomes less incomes as an estimate of risk-neutral values of the GLWB rider.

Theoretically, nested stochastic modeling involves buying and selling equity index futures. The inner loop involves risk-neutral valuation of the GLWB rider. The function \( v \) can be determined by

\[ v = \frac{1}{2} \left( e^{-r(T-x)} + e^{-r(T-x)} \right) \]

subject to boundary conditions:

\[ \text{variable explanation for } f: \]
- \( f \) the number of economic scenarios projection at \( x \) time.
- \( r \) the risk-free rate of interest.
- \( T \) the time to maturity.
- \( \alpha \) the risk aversion factor.
- \( \beta \) the risk-taking factor.

The risk capital is then determined by the average of the 30% worst lowest surpluses/deficiencies from all scenarios form an empirical distribution.

Inner loops:
In each time step of the outer loop, the hedging program is determined by the so-called delta hedging strategy, which involves buying and selling equity index futures. The inner loops determine how much to buy or sell according to the then-current economic conditions.

- Calculate risk-neutral of the GLWB liability at the desired time point under various account values.
- Determine the so-called delta, which is a sensitivity measure of GLWB liability with respect to changes in equity index, by the difference quotient as explained below.

\[ \Delta \text{GLWB} = \frac{f(x+\Delta x) - f(x)}{\Delta x} \]

Monte Carlo simulation may take up to 22.2 years if we run 100 inner loops for risk-neutral valuation for each of the 1000 outer loop scenarios and rebalancing the hedging portfolio 100 times each year for 50 years. Although it is unrealistic to run simulations for such a long period of time, the estimates show the dramatic efficiency of the PDE approach. All computations are done on a personal laptop with an Intel Core i5-6200U CPU (2.30GHz) and 8192MB RAM.

Result and Conclusions
We tested both simulations (both outer loops and inner loops) and the PDE approach (simulations for outer loops and delta calculations through PDEs in inner loops).

It turns out that the risk capital calculation with the PDE method only requires roughly 300 seconds plus additional time for generating outer loop scenarios. In contrast, it is nearly impossible to conduct the same calculations with fully nested simulations. By a rough estimate, we find that running the algorithm may take up to 22.2 years if we run 100 inner loops for risk-neutral valuation for each of the 1000 outer loop scenarios and rebalancing the hedging portfolio 100 times each year for 50 years. Although it is unrealistic to run simulations for such a long period of time, the estimates show the dramatic efficiency of the PDE approach.

We also observed improvements by other non-PDE methods, such as Least Square Monte Carlo (LSMC) and the method of preprocessed inner loop.

References: