NON-REMOVABILITY OF SIERPIŃSKI SPACES

DIMITRIOS NTALAMPEKOS AND JANG-MEI WU

Abstract. We prove that all Sierpiński spaces in $\mathbb{S}^n$, $n \geq 2$, are non-removable for (quasi)conformal maps, generalizing the result of the first named author [Nta18a]. More precisely, we show that for any Sierpiński space $X \subset \mathbb{S}^n$ there exists a homeomorphism $f\colon \mathbb{S}^n \to \mathbb{S}^n$, conformal in $\mathbb{S}^n \setminus X$, that maps X to a set of positive measure and is not globally (quasi)conformal. This is the first class of examples of non-removable sets in higher dimensions.

1. Introduction

In this work we approach the problem of (quasi)conformal removability in $\mathbb{S}^n$ for $n \geq 3$. A compact set $K \subset \mathbb{S}^n$ is *(quasi)conformally removable* if every homeomorphism $f\colon \mathbb{S}^n \to \mathbb{S}^n$ that is (quasi)conformal in $\mathbb{S}^n \setminus K$ is, in fact, (quasi)conformal everywhere. In dimension 2, a set is conformally removable if and only if it is quasiconformally removable; see [You15] for a survey of results. The unavailability of the techniques involving the Beltrami equation in higher dimensions does not allow us to draw such a conclusion, and so far we only have the trivial implication that quasiconformal removability implies conformal removability for sets of measure zero, because 1-quasiconformal mappings are conformal [Geh62, Theorem 15].

Note that there are sets of positive measure that are conformally removable, a phenomenon that does not occur in dimension 2. For example, let $K \subset \mathbb{S}^n$, $n \geq 3$, be a compact set with empty interior and positive measure such that $\mathbb{S}^n \setminus K$ is connected. Then a homeomorphism $f\colon \mathbb{S}^n \to \mathbb{S}^n$ that is conformal on $\mathbb{S}^n \setminus K$ is actually a Möbius map on $\mathbb{S}^n \setminus K$ (by Liouville's Theorem [Geh62, Section 29]), and thus on $\mathbb{S}^n$ by continuity. This implies that K is conformally removable. On the other hand, if $K = C \times [0,1]^{n-1}$, where $C \subset \mathbb{R}$ is a fat Cantor set, then one can show that K is non-removable for quasiconformal maps; see for example the argument in [Nta18b, p. 6]. Since $\mathbb{R}^n \setminus K$ is connected, it follows that K is conformally removable and thus conformal and quasiconformal removability are not equivalent for sets of positive measure in dimensions greater than 2.

The techniques used to prove that a set is (quasi)conformally removable in higher dimensions are the same as the planar ones. In particular, all sets of σ-finite Hausdorff $(n-1)$-measure are removable in $\mathbb{S}^n$ [Väi71, Theorem 35.1], as also are boundaries of domains satisfying a certain quasihyperbolic condition [JS00].

There are very few non-trivial examples of quasiconformally non-removable sets in higher dimensions (e.g. [Bis91]), the main difficulty being the lack of tools for the construction of homeomorphisms with good control of the quasiconformal distortion; in consrast, for planar constructions see [Bis94], [KW96], [Kau84], [Nta18b],

Date: December 14, 2018.
2010 *Mathematics Subject Classification.* Primary 30C65, 57A15; Secondary 54F20.
Key words and phrases. Sierpiński space, Sierpiński carpet, Removability, Conformal maps, Quasiconformal maps.

[Nta18a]. It is not even known whether all sets of positive measure are quasiconformally non-removable. We pose a stronger question (see also [Bis94, Question 3]), known to have a positive answer (partially) in 2 dimensions [KW96]:

Question 1. Let $K \subset \mathbb{S}^n$ $(n \geq 2)$ be a compact set of positive Lebesgue measure. Does there exist a homeomorphism f of $\mathbb{S}^n$ that is quasiconformal on $\mathbb{S}^n \setminus K$ and maps K (or a subset of positive measure) to a set of measure zero?

Note that a positive answer to this question would imply that all sets of positive Lebesgue measure are non-removable for quasiconformal maps.

In this work we prove that a large class of sets, namely Sierpiński spaces, are non-removable for (quasi)conformal maps in $\mathbb{S}^n$. This generalizes the 2-dimensional result from work of the first named author [Nta18a]. Sierpiński spaces are higher-dimensional analogs of planar Sierpiński carpets.

Definition 1.1. A continuum $X \subset \mathbb{S}^n$, $n \geq 2$, is an $(n-1)$-dimensional Sierpiński space if its complement $\mathbb{S}^n \setminus X$ consists of countably many components U_i, $i \in \mathbb{N}$, satisfying the following conditions:

1. $\mathbb{S}^n \setminus U_i$ is an n-cell for each $i \in \mathbb{N}$,
2. $\overline{U_i} \cap \overline{U_j} = \emptyset$ for $i \neq j$,
3. $\bigcup_{i\in\mathbb{N}} U_i$ is dense in $\mathbb{S}^n$, and
4. $\operatorname{diam}(U_i) \to 0$ as $i \to \infty$.

In $\mathbb{S}^n$, $n \geq 3$, the boundary components ∂U_i, $n \in \mathbb{N}$, of $\mathbb{S}^n \setminus X$ are not assumed to be flat spheres by Definition 1.1. In $\mathbb{S}^2$, condition (1) is equivalent to requiring that ∂U_i, $i \in \mathbb{N}$, are Jordan curves, or equivalently flat 1-spheres.

All $(n-1)$-Sierpiński spaces in $\mathbb{S}^n$, for a fixed $n \geq 2$, are homeomorphically equivalent. This topological result was proved by Whyburn for $n = 2$, and by Cannon for dimensions $n = 3$ and $n \geq 5$. Cannon's proof is based on the known Annulus Theorem at that time. Since the Annulus Conjecture has now been proved in dimension 4 (see Theorem 4.3), Cannon's proof extends to $n = 4$ also.

Theorem 1.2 ([Why58], [Can73])**.** *If $X, Y \subset \mathbb{S}^n$, $n \geq 2$, are $(n-1)$-dimensional Sierpiński spaces, then there exists a homeomorphism $f\colon X \to Y$.*

Our main result is the following:

Theorem 1.3. *Let $X \subset \mathbb{S}^n$, $n \geq 2$, be an $(n-1)$-dimensional Sierpiński space. Then there exist a Sierpiński space $Y \subset \mathbb{S}^n$ of positive Lebesgue measure and a homeomorphism $f\colon \mathbb{S}^n \to \mathbb{S}^n$ which maps X onto Y and is conformal on $\mathbb{S}^n \setminus X$.*

The statement is different from the 2-dimensional result in [Nta18a]. It is proved in [Nta18a] that if $X, Y \subset \mathbb{S}^2$ are *any* Sierpiński carpets (i.e., 1-dimensional Sierpiński spaces), then there exists a homeomorphism $f\colon \mathbb{S}^2 \to \mathbb{S}^2$ which maps X onto Y and is conformal on $\mathbb{S}^2 \setminus X$. We do not expect such a strong statement in higher dimensions. Firstly, the boundary components ∂U_i, $i \in \mathbb{N}$, of the complement of a Sierpiński space are not necessarily flat spheres. Secondly, even under the stronger assumption that all ∂U_i, $i \in \mathbb{N}$, are flat $(n-1)$-spheres in $\mathbb{S}^n$, topological open balls U_i, $i \in \mathbb{N}$, need not be quasiconformally equivalent to an open Euclidean ball [GV65] – in contrast to the 2-dimensional case in which the Riemann Mapping Theorem can be invoked.

The proof of Theorem 1.3 follows the lines of Whyburn and Cannon. However, in order to prove the (quasi)conformal non-removability of an $(n-1)$-Sierpiński space

X in $\mathbb{S}^n$, we are not allowed to alter the topology of the complementary components of X in $\mathbb{S}^n$, but we can only use (quasi)conformal deformations of them. For this reason, we use a decomposition of $\mathbb{S}^n$ whose degenerate elements are not necessarily n-cells; see Lemma 3.1. This entails some technical complications.

Corollary 1.4. *All $(n-1)$-dimensional Sierpiński spaces in $\mathbb{S}^n$, $n \geq 2$, are non-removable for (quasi)conformal maps.*

Since a (quasi)conformal map f of $\mathbb{S}^n$ necessarily maps sets of measure zero to sets of measure zero [Väi71, Theorem 33.2], this Corollary follows immediately from Theorem 1.3 for Sierpiński spaces having zero measure. If a Sierpiński space has positive measure, then the proof of the non-removability requires an extra ingredient and it is given at the end of Section 2.

We give the proof of Theorem 1.3 in Section 2, based on a topological lemma (Lemma 2.2) proved in Section 3. Finally, Section 4 contains several topological facts that are used throughout the paper.

Acknowledgments: Part of this research was conducted while the first named author was visiting University of Illinois Urbana-Champaign. He thanks the faculty and the staff of the Department of Mathematics for their hospitality. The research of the second named author is partially supported by Simons Foundation Collaboration Grant #353435.

2. Proof of main result

Definition 2.1. Let $\varepsilon > 0$, and Y be a subset of $\mathbb{S}^n$. A partition $\{Y_\#\}$ of Y is called an *ε-subdivision* if there exists a finite cubical complex K on Y whose n-cubes are precisely $\{Y_\#\}$, each of which has diameter less than ε.

A partition $\{X_\#\}$ of an $(n-1)$-dimensional Sierpiński space $X \subset \mathbb{S}^n$ into Sierpiński spaces is an *ε-subdivision (rel. $U_1, \ldots, U_N$)* if there exist components $U_1, \ldots, U_N$ of $\mathbb{S}^n \setminus X$ and an ε-subdivision $\{Y_\#\}$ of the closed domain $Y \coloneqq \mathbb{S}^n \setminus \bigcup_{j=1}^N U_j$, for which

(1) the $(n-1)$-skeleton of the cubical complex associated to the subdivision $\{Y_\#\}$ is contained in $X \setminus \bigcup_{j=N+1}^\infty \overline{U_j}$, and
(2) $\{X_\#\} = \{X \cap Y_\#\}$, i.e., the Sierpiński spaces in the subdivision $\{X_\#\}$ arise as the intersections of the n-cubes in the subdivision $\{Y_\#\}$ with the original Sierpiński space X.

In this case, we say that the subdivisions $\{X_\#\}$ and $\{Y_\#\}$ are *in correspondence.*

We now state the key lemma for the proof of Theorem 1.3.

Lemma 2.2. *Let $n \geq 2$, and X be an $(n-1)$-dimensional Sierpiński space in $\mathbb{S}^n$. Let $\varepsilon > 0$, and $U_1, \ldots, U_N$ be a collection of components in $\mathbb{S}^n \setminus X$ for which the remaining components have diameters less than ε, and let $G_1, \ldots, G_N \subset \mathbb{S}^n$ be open sets with pairwise disjoint closures for which $\mathbb{S}^n \setminus G_1, \ldots, \mathbb{S}^n \setminus G_N$ are n-cells. Given N orientation-preserving homeomorphisms $h_j \colon \partial U_j \to \partial G_j$, $j \in \{1, \ldots, N\}$, there exist a homeomorphism*

$$h \colon \mathbb{S}^n \setminus \bigcup_{j=1}^N U_j \to \mathbb{S}^n \setminus \bigcup_{j=1}^N G_j$$

which extends h_j, $j \in \{1,\dots,N\}$, and ε-subdivisions of X and $\mathbb{S}^n \setminus \bigcup_{j=1}^N G_j$, respectively, which are in correspondence under h.

The proof of this lemma is given in the next section.

Proof of Theorem 1.3. Let $\varepsilon_k = 1/k$, and fix a component U_0 of $\mathbb{S}^n \setminus X$ that has the largest diameter.

First let $\varepsilon = \varepsilon_1$, and $U_0, U_1, \dots, U_N$ be a collection of components in $\mathbb{S}^n \setminus X$ for which the remaining components have diameters less than ε_1. Set $G_0 \coloneqq U_0$, and let h_1 be an embedding of $\bigcup_{j=0}^N \overline{U_j}$ into $\mathbb{S}^n$, which is the identity map on $\overline{U_0}$, and is a similarity on $\overline{U_j}$ that shrinks and translates U_j to a set G_j for each $j \in \{1,\dots,N\}$, so that the sets $\overline{G_j}$ are disjoint subsets of $\mathbb{S}^n \setminus \overline{U_0}$. Then, by Lemma 2.2, there exist a global homeomorphic extension $h_1 \colon \mathbb{S}^n \to \mathbb{S}^n$, an ε_1-subdivision $\{X_\#\}$ of X and an ε_1-subdivision $\{Z_\#\}$ of $Z \coloneqq \mathbb{S}^n \setminus \bigcup_{j=0}^N G_j$, respectively, for which $\{X_\#\}$ and $\{h_1^{-1}(Z_\#)\}$ are in correspondence.

In the second step, let $X_\#$ be a Sierpiński space in the ε_1-subdivision of X, and $Z_\#$ be the corresponding n-cell in the ε_1-subdivision of Z under h_1^{-1}. Let V_0 be the complementary component of $X_\#$ that contains U_0, and fix a finite collection of components $V_0, V_1, \dots, V_M$ in $\mathbb{S}^n \setminus X_\#$ for which all remaining components have diameters less than ε_2. We observe that $\mathbb{S}^n \setminus V_0 = h_1^{-1}(Z_\#)$. Again, by Lemma 2.2, there exist a homeomorphism h_2 from $\mathbb{S}^n \setminus V_0$ onto $Z_\#$, and ε_2-subdivisions of $X_\#$ and $Z_\#$, respectively, for which

(1) h_2 agrees with h_1 on ∂V_0, and shrinks and translates $V_1, \dots, V_M$ to sets $D_1, \dots, D_M$, having pairwise disjoint closures, in the interior of $Z_\#$ by similarities, and
(2) the ε_2-subdivision of $X_\#$ and the preimage under h_2 of the ε_2-subdivision of $Z_\#$ are in correspondence.

We repeat this for each Sierpiński space $X_\#$ in the ε_1-subdivision of X, and extend h_2 to a homeomorphism $\mathbb{S}^n \to \mathbb{S}^n$ which agrees with h_1 on $\bigcup_{j=0}^N \overline{U_j}$.

Inductively, we obtain a sequence of homeomorphisms $h_k \colon \mathbb{S}^n \to \mathbb{S}^n$ for which h_k is ε_k-close to h_m uniformly for all $m \geq k$. The same statement holds for the inverses h_k^{-1}. In view of the inductive construction, on each component of $\mathbb{S}^n \setminus X$, the sequence $\{h_k\}_{k\in\mathbb{N}}$ is eventually constant and, in fact, the maps are eventually conformal. Indeed, if U is a component of $\mathbb{S}^n \setminus X$ with $\varepsilon_k \leq \operatorname{diam}(U)$, then $h_k|_U$ is a similarity and $h_m|_U = h_k|_U$ for all $m \geq k$.

We may conclude that the sequence h_k converges uniformly to a homeomorphism $h \colon \mathbb{S}^n \to \mathbb{S}^n$ which has the following properties:

(a) h is conformal in the complement of X,
(b) $Y \coloneqq h(X)$ is a Sierpiński space, and
(c) Y has positive n-measure.

For the latter property one has to note that the image of $\mathbb{S}^n \setminus X$ under h has smaller Lebesgue measure, since h is a similarity on each component of $\mathbb{S}^n \setminus X$ and it shrinks all but one component. In fact, h may be chosen so that the measure of Y is arbitrarily close to that of $\mathbb{S}^n$. □

Proof of Corollary 1.4. Identify the Sierpiński space X with a compact subset of $\mathbb{R}^n \subset \mathbb{R}^n \cup \{\infty\} \approx \mathbb{S}^n$ by the stereographic projection, and let U_0 be the unbounded component of $\mathbb{R}^n \setminus X$. Enumerate the remaining complementary components of X by U_i, $i \in \mathbb{N}$.

By the preceding proof, we may obtain a homeomorphism h of $\mathbb{R}^n$ which is the identity on U_0 and is, for each $i \in \mathbb{N}$, a similarity on U_i that shrinks U_i by any desired factor. In particular, we may require that $|h(U_i)| \le |U_i|^{i+1}$ for $i \in \mathbb{N}$, where $|\cdot|$ denotes Lebesgue measure in $\mathbb{R}^n$. Note that the Jacobian of h on U_i is a constant equal to $|h(U_i)|/|U_i|$.

If h^{-1} were quasiconformal, then its Jacobian $J_{h^{-1}}$ would be in $L^{1+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $\varepsilon > 0$ depending on h^{-1}; see [Geh73, Theorem 1]. On the other hand,

$$\begin{aligned}\int_{\bigcup_{i\in\mathbb{N}} h(U_i)} (J_{h^{-1}})^{1+\varepsilon} &= \sum_{i\in\mathbb{N}} \frac{|U_i|^{1+\varepsilon}}{|h(U_i)|^{1+\varepsilon}} |h(U_i)| \\ &= \sum_{i\in\mathbb{N}} |U_i|^{1+\varepsilon} |h(U_i)|^{-\varepsilon} \ge \sum_{i\in\mathbb{N}} |U_i|^{1-\varepsilon i},\end{aligned}$$

which diverges because $\operatorname{diam}(U_i) \to 0$. This contradiction proves that h^{-1}, and thus h, is not quasiconformal on $\mathbb{R}^n$. □

3. Proof of Lemma 2.2

Under the assumptions of Lemma 2.2, we first prove a decomposition result suitable for our setting (compare to the statement of Theorem 4.1):

Lemma 3.1. *Under the assumptions of Lemma 2.2, there exists a continuous surjective map* $p\colon \mathbb{S}^n \to \mathbb{S}^n$ *which fixes* $\bigcup_{i=1}^N \overline{U_j}$*, and induces a decomposition of* $\mathbb{S}^n$ *into sets* $\{\overline{U_j}\}_{j\ge N+1}$ *and points. In particular, there exist countably many points* $\{q_j : j \ge N+1\}$ *for which* $p^{-1}(q_j) = \overline{U_j}$*, and the map* $p\colon \mathbb{S}^n \setminus \bigcup_{j=N+1}^\infty \overline{U_j} \to \mathbb{S}^n \setminus \{q_j : j \ge N+1\}$ *is bijective.*

Proof. Consider first an embedding $f\colon X \to \mathbb{S}^n$ such that boundary components of the complement of $X' := f(X)$ in $\mathbb{S}^n$ are flat $(n-1)$-spheres. Such a map exists by Lemma 4.2. We remark that each sphere $f(\partial U_j)$ bounds a complementary component, denoted by U_j', of X', and that $X' = \mathbb{S}^n \setminus \bigcup_{j=1}^\infty U_j'$. For an explanation of this remark, see for example the argument in [Bon11, Lemma 5.5].

Consider the homeomorphisms $g_j = (f|_{\partial U_j})^{-1}\colon \partial U_j' \to \partial U_j$, $j = 1, \dots, N$. By Corollary 4.6, these maps may be extended to a homeomorphism $g\colon \mathbb{S}^n \setminus \bigcup_{j=1}^N U_j' \to \mathbb{S}^n \setminus \bigcup_{j=1}^N U_j$. Note that g maps each $\overline{U_j'}$ to a flat n-cell, for $j \ge N+1$.

Consider the homeomorphism $G = g \circ f$ on X. Since G is the identity on $\bigcup_{j=1}^N \partial U_j$, it may be extended to be the identity map on $\overline{\Omega}$, where $\Omega := \bigcup_{j=1}^N U_j$. Moreover, for $j \ge N+1$, G maps ∂U_j to a flat sphere that bounds a complementary component, denoted by U_j'', of $X'' = G(X)$.

We apply the Decomposition Theorem 4.1 to obtain a map $p''\colon \mathbb{S}^n \to \mathbb{S}^n$ that fixes $\overline{\Omega}$, and collapses $\overline{U_j''}$, $j \ge N+1$, to points. The composition

$$p := p'' \circ G\colon X \cup \Omega \to \mathbb{S}^n$$

is the identity on $\overline{\Omega}$, and collapses ∂U_j to a point q_j satisfying $p^{-1}(q_j) = \partial U_j$ for $j \ge N+1$. We now extend p to $\bigcup_{j=N+1}^\infty U_j$ so that $p^{-1}(q_j) = \overline{U_j}$. Since $\operatorname{diam}(U_j) \to 0$, the extension $p\colon \mathbb{S}^n \to \mathbb{S}^n$ is continuous, and hence is the map claimed in the lemma. □

Let $p\colon \mathbb{S}^n \to \mathbb{S}^n$ be the map in Lemma 3.1. By Corollary 4.6, the homeomorphisms

$$h_j \circ (p|_{\partial U_j})^{-1}\colon p(\partial U_j) \to \partial G_j, \quad 1 \leq j \leq N,$$

on subsets of $\mathbb{S}^n$, can be extended to a homeomorphism

$$H\colon p(\mathbb{S}^n) \setminus \bigcup_{j=1}^{N} p(U_j) \to \mathbb{S}^n \setminus \bigcup_{j=1}^{N} G_j.$$

Fix a number $\delta \in (0,\varepsilon)$, for which all sets of diameter δ are mapped by $p^{-1} \circ H^{-1}$ to sets of diameters less than ε. The existence of such a δ is a consequence of the uniform continuity of H^{-1}, p and the fact that the preimages of points under p have diameter smaller than ε; recall from the statement of Lemma 2.2 that the sets $\overline{U_j}$, $j \geq N+1$, that are collapsed to points have diameters smaller than ε.

Fix next a finite cubical complex K on $\mathbb{S}^n \setminus \bigcup_{j=1}^N G_j$ whose n-cubes have diameters less than δ, and whose $(n-1)$-skeleton $K^{[n-1]}$ does not meet the countable set $A \coloneqq \{H(q_j)\colon j \geq N+1\}$. The cubical complex K may be found by identifying $\mathbb{S}^n$ with $\mathbb{R}^n \cup \{\infty\}$ and $\{G_j\colon 1 \leq j \leq N\}$ with Euclidean cubes having edges parallel to coordinate axes, with the help of Corollary 4.6. Then, the family of n-cells $\{C\colon C \text{ is an } n\text{-cube in } K\}$ is a δ-subdivision of $Z \coloneqq \mathbb{S}^n \setminus \bigcup_{j=1}^N G_j$.

Observe that $H \circ p\colon \mathbb{S}^n \setminus \bigcup_{j=1}^N U_j \to \mathbb{S}^n \setminus \bigcup_{j=1}^N G_j$ is a cellular map between two compact manifolds *with boundary*. For the purpose of applying the Approximation Theorem (Corollary 4.8), we attach, on the domain side, an n-cell C_j to $\mathbb{S}^n \setminus \bigcup_{j=1}^N U_j$ along ∂U_j for every $j = 1, \dots, N$, to obtain an expanded space $\widetilde{M}$ which is homeomorphic to $\mathbb{S}^n$, and we do the same on the target side to obtain an expanded space $\widetilde{N}$ homeomorphic to $\mathbb{S}^n$. We extend $H \circ p$ to a map $\widetilde{M} \to \widetilde{N}$, which is a homeomorphism between every pair of added n-cells. We now apply Corollary 4.8 to conclude that $\{(H \circ p)^{-1}(C)\colon C \text{ is an } n\text{-cube in } K\}$ are n-cells, and that they form an ε-subdivision of $Y \coloneqq \mathbb{S}^n \setminus \bigcup_{j=1}^N U_j$. Thus, $\{X \cap (H \circ p)^{-1}(C)\colon C \text{ is an } n\text{-cube in } K\}$ is an ε-subdivision of X.

Observe that $(H \circ p)^{-1}$ is injective on the space $|K^{[n-1]}|$ of the $(n-1)$-skeleton $K^{[n-1]}$, since the latter does not meet the set A. Set $L = (H \circ p)^{-1}(|K^{[n-1]}|)$. The homeomorphism h in Lemma 2.2 may be obtained by first setting

$$h|_L = H \circ p|_L,$$

and then extending $h|_L$ to the interior of the n-cells in the ε-subdivision of Y homeomorphically. This completes the proof of Lemma 2.2.

4. Topological Facts

We record some topological facts that are needed for the proof of Lemma 2.2. We state them and provide references for $n \geq 3$, but all these statements are also true for $n = 2$. We refer the reader to [DV09] for the definitions of the various topological notions appearing below.

Theorem 4.1 (Decomposition Theorem, [Moo25], [Mey63], [Dav86, II.8.6A])**.** *Let $n \geq 2$, $\{B_i\}_{i\in\mathbb{N}}$ be a null sequence of disjoint flat n-cells in $\mathbb{S}^n$, and U be an open set containing $\bigcup_{i\in\mathbb{N}} B_i$. Then there exists a continuous surjective map $f\colon \mathbb{S}^n \to \mathbb{S}^n$, which is the identity outside U, such that f induces a decomposition of $\mathbb{S}^n$ into the sets $\{B_i\}_{i\in\mathbb{N}}$ and points. In particular, there exist countably many points $p_i \in U$,*

$i \in \mathbb{N}$, such that $f^{-1}(p_i) = B_i$ for each $i \in \mathbb{N}$, and the map $f\colon \mathbb{S}^n \setminus \bigcup_{i\in\mathbb{N}} B_i \to \mathbb{S}^n \setminus \{p_i : i \in \mathbb{N}\}$ is bijective.

Lemma 4.2 ([Can73, Lemma 0]). *Let $n \geq 3$, and X be an $(n-1)$-dimensional Sierpiński space in $\mathbb{S}^n$. Then there exists an embedding $h\colon X \to \mathbb{S}^n$ such that the boundary components of $\mathbb{S}^n \setminus h(X)$ are flat $(n-1)$-spheres.*

Theorem 4.3 (Annulus Theorem, [Moi52], [Kir69], [Qui82]). *Let $n \geq 3$, and $D_1, D_2 \subset \mathbb{S}^n$ be disjoint flat n-cells. Then $\mathbb{S}^n \setminus (D_1 \cup D_2)$ is homeomorphic to $\mathbb{S}^{n-1} \times [0,1]$.*

The following is a consequence of the Annulus Theorem; see Kirby [Kir69].

Theorem 4.4 (Isotopy Theorem). *Let $n \geq 3$, and $f\colon \mathbb{S}^n \to \mathbb{S}^n$ be an orientation-preserving homeomorphism. Then f is isotopic to the identity.*

The following extension property follows from the Annulus Theorem almost immediately. We state it here for completeness.

Proposition 4.5 (Extension). *Let $n \geq 3$, $U_1, \ldots, U_N \subset \mathbb{S}^n$ be open sets with pairwise disjoint closures and whose boundaries are flat $(n-1)$-spheres, and let $U_1', \ldots, U_N' \subset \mathbb{S}^n$ be another collection of open sets with the same properties. Let $h_i\colon \partial U_i \to \partial U_i'$, $i \in \{1, \ldots, N\}$, be orientation-preserving homeomorphisms. Then there exists a homeomorphism*

$$h\colon \mathbb{S}^n \to \mathbb{S}^n$$

which extends h_i for $i \in \{1, \ldots, N\}$.

Proof. We prove by induction on the number N of open sets in each collection. The statement is true for $N = 1$, by the flatness.

Suppose $N = 2$. By the Annulus Theorem (Theorem 4.3), there exist homeomorphisms

$$\varphi\colon \mathbb{S}^n \setminus (U_1 \cup U_2) \to \mathbb{S}^{n-1} \times [0,1] \quad \text{and} \quad \varphi'\colon \mathbb{S}^n \setminus (U_1' \cup U_2') \to \mathbb{S}^{n-1} \times [0,1].$$

Next, by the Isotopy Theorem (Theorem 4.4), there exists a homeomorphism $F\colon \mathbb{S}^{n-1} \times [0,1] \to \mathbb{S}^{n-1} \times [0,1]$ which is an isotopy between $\varphi' \circ h_0 \circ \varphi^{-1}|_{\mathbb{S}^{n-1}\times\{0\}}$ and $\varphi' \circ h_1 \circ \varphi^{-1}|_{\mathbb{S}^{n-1}\times\{1\}}$. Then the homeomorphism $h \coloneqq {\varphi'}^{-1} \circ F \circ \varphi\colon \mathbb{S}^n \setminus (U_1 \cup U_2) \to \mathbb{S}^n \setminus (U_1' \cup U_2')$ extends $h_1|_{\partial U_1}$ and $h_2|_{\partial U_2}$, and h may be extended homeomorphically to a map $\mathbb{S}^n \to \mathbb{S}^n$ as claimed in the proposition.

Consider now the case when there are N open sets in each collection, and N boundary homeomorphisms $h_i\colon \partial U_i \to \partial U_i'$, $i \in \{1, \ldots, N\}$. Assume, by the induction hypothesis, that the proposition has been proved when the number is $N-1$. Fix now a homeomorphism $g\colon \mathbb{S}^n \to \mathbb{S}^n$ which extends $h_i\colon \partial U_i \to \partial U_i'$ for $i \in \{1, \ldots, N-1\}$.

We claim that there is a flat n-cell D which contains $\overline{U_N'}$ and $g(\overline{U_N})$ in its interior and keeps $\bigcup_{i=1}^{N-1} \overline{U_i'}$ in its complement.

To this end, one can first convert all cells $\overline{U_i'}$, $i = 1, \ldots, N-1$, to round balls, by the induction assumption, with a global homeomorphism of $\mathbb{S}^n$. Thus, we suppose that $\overline{U_i'}$, $i = 1, \ldots, N-1$, are round balls. Fix next $N-2$ pairwise disjoint flat n-cells, $L_1, \ldots, L_{N-2}$, contained in

$$W \coloneqq \mathbb{S}^n \setminus \left(\overline{U_N'} \cup g(\overline{U_N}) \cup \bigcup_{i=1}^{N-1} U_i' \right),$$

which have the properties that (a) $L_i \cap \overline{U_i'}$ and $L_i \cap \overline{U_{i+1}'}$ are flat $(n-1)$-cells in $\partial U_i'$ and $\partial U_{j+1}'$, respectively, and $L_i \cap \overline{U_j'} = \emptyset$ for $j \neq i, i+1$, (b) the set $E \coloneqq \overline{U_1'} \cup L_1 \cup \cdots \cup \overline{U_{N-2}'} \cup L_{N-2} \cup \overline{U_{N-1}'}$ is a flat n-cell, and (c) the set $\overline{\mathbb{S}^n \setminus E}$ is also a flat n-cell which contains $\overline{U_N'} \cup g(\overline{U_N})$ in its interior. Each of these n-cells L_i can be constructed by connecting the balls $\overline{U_i'}$ and $\overline{U_{i+1}'}$ with a smooth path in W and then fattening the path to obtain a smooth cylinder L_i. By shrinking the flat cell $\overline{\mathbb{S}^n \setminus E}$ we may obtain a flat n-cell $D \subset \mathbb{S}^n \setminus E$ with the claimed properties.

Applying the initial step (for $N = 2$) to the region $D \setminus g(U_N)$, we obtain a homeomorphism $f \colon D \setminus g(U_N) \to D \setminus U_N'$ which agrees with the homeomorphisms $h_N \circ g^{-1}|_{\partial(g(U_N))}$ and $\mathrm{id}|_{\partial D}$ on the boundary. The map $h = f \circ g \colon \mathbb{S}^n \setminus \bigcup_{i=1}^N U_i \to \mathbb{S}^n \setminus \bigcup_{i=1}^N U_i'$, satisfying $h|_{\partial U_i} = h_i$, may then be extended to a homeomorphism $\mathbb{S}^n \to \mathbb{S}^n$. □

Corollary 4.6. *Let $n \geq 3$, $U_1, \dots, U_N \subset \mathbb{S}^n$ be open sets having pairwise disjoint closures and for which $\mathbb{S}^n \setminus U_1, \dots, \mathbb{S}^n \setminus U_N$ are n-cells, and let $U_1', \dots, U_N' \subset \mathbb{S}^n$ be another collection of open sets with the same properties. Let $h_i \colon \partial U_i \to \partial U_i'$, $i \in \{1, \dots, N\}$, be orientation-preserving homeomorphisms. Then there exists a homeomorphism*

$$h \colon \mathbb{S}^n \setminus \bigcup_{i=1}^N U_i \to \mathbb{S}^n \setminus \bigcup_{i=1}^N U_i'$$

which extends h_i for $i \in \{1, \dots, N\}$.

Corollary 4.6 follows from Proposition 4.5 as follows. Following the proof of Lemma 0 in [Can73], we can "enlarge" the complementary components U_i, $i \in \{1, \dots, N\}$, slightly by an embedding $\psi \colon \mathbb{S}^n \setminus \bigcup_{i=1}^N U_i \hookrightarrow \mathbb{S}^n \setminus \bigcup_{i=1}^N U_i$ in such a way that the boundary components $\psi(\partial U_i)$ of the new regions are flat $(n-1)$-spheres. We do the same for $\mathbb{S}^n \setminus \bigcup_{i=1}^N U_i'$ by an embedding ψ'. We apply Proposition 4.5 to the new regions to find a homeomorphic extension, and then use ψ^{-1} and ψ'^{-1} to pull the extension back to the original regions.

We also need the following approximation theorem for cell-like and cellular maps; recall that cellular maps are cell-like.

Theorem 4.7 (Approximation Theorem for Cell-like/Cellular Maps)**.** *Let $f \colon M \to N$ be a cell-like map between n-manifolds when $n \geq 4$, or a cellular map when $n = 3$. Then f is a near-homeomorphism.*

In particular, suppose that ρ is a metric on N, C is a closed subset of N for which $f|_{f^{-1}(C)}$ is injective, and $\epsilon \colon N \to [0, \infty)$ is a continuous function satisfying $\epsilon^{-1}(0) = C$. Then there is a homeomorphism $g \colon M \to N$ satisfying $\rho(g(x), f(x)) < \epsilon(f(x))$ for all $x \in M \setminus f^{-1}(C)$ and $g|_{f^{-1}(C)} = f|_{f^{-1}(C)}$.

Theorem 4.7 was proved for dimension 3 in [Arm71], for dimension $n \geq 5$ in [Sie72], and for dimension 4 in [Anc86]. The version above is stated in Corollary 7.4.3 in [DV09]. For our application, we need the following corollary.

Corollary 4.8. *Let $f \colon M \to N$ be a cellular map between n-manifolds, where $n \geq 3$. Let $B \subset N$ be an n-cell and $C = \partial B$. Suppose that f is injective on $f^{-1}(C)$. Then $f^{-1}(B)$ is an n-cell in M, whose boundary is $f^{-1}(C)$.*

References

[Anc86] F. D. Ancel, *Resolving wild embeddings of codimension-one manifolds in manifolds of dimensions greater than* 3, Topology Appl. **24** (1986), no. 1-3, 13–40. Special volume in honor of R. H. Bing (1914–1986).

[Arm71] S. Armentrout, *Cellular decompositions of* 3*-manifolds that yield* 3*-manifolds*, American Mathematical Society, Providence, R. I., 1971. Memoirs of the American Mathematical Society, No. 107.

[Bis91] C. J. Bishop, *Non-removable sets for quasiconformal and bilipschitz mappings in* $\mathbb{R}^3$, Stony Brook IMS preprint (1991).

[Bis94] ______, *Some homeomorphisms of the sphere conformal off a curve*, Ann. Acad. Sci. Fenn. Ser. A I Math. **19** (1994), no. 2, 323–338.

[Bon11] M. Bonk, *Uniformization of Sierpiński carpets in the plane*, Invent. Math. **186** (2011), no. 3, 559–665.

[Can73] J. W. Cannon, *A positional characterization of the* $(n-1)$*-dimensional Sierpiński curve in* $S^n (n \neq 4)$, Fund. Math. **79** (1973), no. 2, 107–112.

[Dav86] R. J. Daverman, *Decompositions of manifolds*, Pure and Applied Mathematics, vol. 124, Academic Press, Inc., Orlando, FL, 1986.

[DV09] R. J. Daverman and G. A. Venema, *Embeddings in manifolds*, Graduate Studies in Mathematics, vol. 106, American Mathematical Society, Providence, RI, 2009.

[Geh62] F. W. Gehring, *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. **103** (1962), 353–393.

[Geh73] F. W. Gehring, *The* L^p*-integrability of the partial derivatives of a quasiconformal mapping*, Acta Math. **130** (1973), 265–277.

[GV65] F. W. Gehring and J. Väisälä, *The coefficients of quasiconformality of domains in space*, Acta Math. **114** (1965), 1–70.

[JS00] P. W. Jones and S. K. Smirnov, *Removability theorems for Sobolev functions and quasiconformal maps*, Ark. Mat. **38** (2000), no. 2, 263–279.

[Kau84] R. Kaufman, *Fourier-Stieltjes coefficients and continuation of functions*, Ann. Acad. Sci. Fenn. Ser. A I Math. **9** (1984), 27–31.

[Kir69] R. C. Kirby, *Stable homeomorphisms and the annulus conjecture*, Ann. of Math. (2) **89** (1969), 575–582.

[KW96] R. Kaufman and J.-M. Wu, *On removable sets for quasiconformal mappings*, Ark. Mat. **34** (1996), no. 1, 141–158.

[Mey63] D. V. Meyer, *A decomposition of* E^3 *into points and a null family of tame* 3*-cells is* E^3, Ann. of Math. (2) **78** (1963), 600–604.

[Moi52] E. E. Moise, *Affine structures in* 3*-manifolds. V. The triangulation theorem and Hauptvermutung*, Ann. of Math. (2) **56** (1952), 96–114.

[Moo25] R. L. Moore, *Concerning upper semi-continuous collections of continua*, Trans. Amer. Math. Soc. **27** (1925), no. 4, 416–428.

[Nta18a] D. Ntalampekos, *Non-removability of Sierpiński carpets*, preprint arXiv:1809.05605, 2018.

[Nta18b] ______, *Non-removability of the Sierpiński Gasket*, preprint arXiv:1804.10239, 2018.

[Qui82] F. Quinn, *Ends of maps. III. Dimensions* 4 *and* 5, J. Differential Geom. **17** (1982), no. 3, 503–521.

[Sie72] L. C. Siebenmann, *Approximating cellular maps by homeomorphisms*, Topology **11** (1972), 271–294.

[Väi71] J. Väisälä, *Lectures on* n*-dimensional quasiconformal mappings*, Lecture Notes in Mathematics, vol. 229, Springer-Verlag, Berlin-New York, 1971.

[Why58] G. T. Whyburn, *Topological characterization of the Sierpiński curve*, Fund. Math. **45** (1958), 320–324.

[You15] M. Younsi, *On removable sets for holomorphic functions*, EMS Surv. Math. Sci. **2** (2015), no. 2, 219–254.

Institute for Mathematical Sciences, Stony Brook University, Stony Brook, NY 11794, USA.
E-mail address: dimitrios.ntalampekos@stonybrook.edu

Department of Mathematics, University of Illinois, Urbana, IL 61822, USA.
E-mail address: jmwu@illinois.edu