Review on Capital Allocation Principles

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Supervised by Dr. Ying Wang

Haoen Cui
Ziran Gao
Yeon Ju Kim
Yunan Shi

Department of Mathematics

University of Illinois at Urbana-Champaign

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Haoen Cui ∗Ziran Gao †Yeon Ju Kim ‡Yunan Shi §Ying Wang ¶

Abstract In this paper, we review the metrics, ratios and models for the evaluation of capital allocation methods. In addition, we compared the capital allocation with asset allocation from the perspective of their goals and applications. Moreover, we review the unified capital allocation principles. Finally, we propose the capital allocation principles with sub-business lines based on Cai and Wang (2016)'s.

Keywords: Capital allocation principles, EVA, RORAC, RAROC, RARORAC, RBC, add on and off, capital surplus risk, capital deficit risk, policy limits and deductibles

∗Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA (email: hcui10@illinois.edu).
†Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA (email: zgao26@illinois.edu).
‡Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA (email: ykim197@illinois.edu).
§Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA (email: shi38@illinois.edu).
¶Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA (email: wyactsc@illinois.edu).
1 Introduction

Capital allocation plays an essential role in the insurance industry because most insurance policies are purchased not as an investment but to protect the policyholders against insolvency risk. Capital is held to assure policyholders that claims will be paid even if it is larger than expected. Even though the firm does not go bankrupt line by line, to allocate capital to each line of business is of importance to maximize shareholders’ value, which is to increase the market value of equity capital. And the firm should measure its performance by line of business to determine whether each line of business is contributing sufficiently to profits to cover its cost of capital and then create value to the firm.

Capital allocation means the process of distributing the capital to different business lines or portfolio elements. It describes how businesses divide their financial resources and other sources of capital to different processes, people and projects. Overall, it is management’s goal to optimize capital allocation so that it generates as much wealth as possible for its shareholders. The process behind making a capital allocation decision is complex, as management virtually has an unlimited number of options to consider. Moreover, we should identify the different definitions for capital types, which will be illustrated in Section 2.

It should be evaluated both internally and externally since the capital/reserve should be solvent according to the regulation requirements, NAIC, Solvency II, GAAP, LICAT, OSFI and IFRS, and the assets should obtain enough level of liquidity to satisfy adequacy to meet with the payments for liabilities, the benefits/pension payments. In this case, capital allocation principles has to be linked with asset liability management and liquidity risk management. In practice, risk metrics, ratios and models are employed to model risk and determine appropriate amount of capital to be allocated to the special business line. Economic value adjusted, Value-at-Risk, Risk based capital, and Risk-Adjusted Return on Risk-adjusted capital will introduce in Section 3 are metrics or ratios popularly utilized. Metrics, such as ratios and models are evaluation approaches which can be adopted after the allocation of capital and measure the efficiency and profitability of each business line. Hence, they will provide suggestions for future investment or business strategies as well as RBC ratio by (2.4).

In Section 4, we will compare the theories of capital allocation with asset allocation from the perspective of their goals and approaches. Capital asset pricing model and Black-Litterman model are two main model we mainly use for asset allocation. Since the standard mean-variance portfolio optimization approach is sensitive to small changes and impossible to
create reasonable portfolio mix, the Black-Litterman model was created considering both the investor views and market equilibrium.

In the literature, capital allocation principles concerns more about the risk and capital budget before the allocation of capital. The purpose of capital allocation is to benefit the shareholder and to increase the value of company, which is a trade-off of minimizing risks and maximizing profit. Capital allocation is to determine how much money should be allocated to each action or each line of business to yield the most benefit, or rate of return. One of the classical model is by Cheung (2007) in Section 5, and is investigated how to allocate policy limits and deductibles into various business lines.

Another type of model is to allocate the total budgeted capital into different business lines. Denault (2001) recognizes capital allocation principles as a multiple vector of risk measures and defined three properties (no undercut, symmetry and riskless allocation) for a coherent allocation principle should satisfy. In these models, we suppose there are \( n \) business lines for a company. The basic framework of the proportional capital allocation is to first define \( \alpha_j \) as the proportion of the company’s equity capital allocated to business \( j \) \((0 < X_j < 1)\). Let \( C_j \) be the amount of capital allocated to business \( j \), which equals the total capital \( C \) multiplies by \( \alpha_j \). If the firm has \( n \) business, then \( \sum_{j=1}^{n} \alpha_j \leq 1 \) and \( \sum_{j=1}^{n} C_j \leq C \).

Obviously, proportional methods are easy to be understood and calculated. The cons of these principles is that they only consider positive proportions, which means they do not allow for withdrawals from the business lines (the proportions cannot be negative numbers), and it is not beneficial in some situations for the company. In Section 6.1, we will introduce the non-proportional capital allocation principles considering mitigate both the capital surplus and deficit risks from each business line.

For various companies, they may have different sub-business lines that still needs capital sufficiency, and so we will generalize all of the model in Section 6.2 to models with sub-business lines.
2 Capital Categories

There are various categories for capital in the literature, such as risk capital, economic capital and Regulatory Risk-Based Capital.

2.1 Risk Capital

Definition of Risk Capital  Risk capital consists of investment funds allocated to speculative activity and refers to the funds used for high-risk, high-reward investments such as junior mining or emerging biotechnology stocks. Such capital can either earn spectacular returns over a period of time, or it may dwindle to a fraction of the initial amount invested if several ventures prove unsuccessful, so diversification is key for successful investment of risk capital. In the context of venture capital, risk capital may also refer to funds invested in a promising startup.

Risk Capital is the smallest amount that can be invested to insure the value of the firm’s net assets against a loss in value relative to the risk-free investment of those nest assets. It refers to funds invested speculatively in high-risk and high-return instruments, especially a startup. Such instruments can be an investment in emerging stock or even one in oil extraction. Risk Capital usually come from private equity and are funds used in exchange for the opportunity to generate a profitable outside gain.

Characteristics of Risk Capital

- Risk Capital is a type of private equity. Venture Capital (VC) is the most common type of private equity.

- Risk and return are usually positively correlated. An investment with a higher risk usually generates a higher rate of return, vice versa. If an investor is risk averse, then a lower proportion of risk capital should be allocated in the total portfolio. A junior mining, for example, may bring new opportunities for investors before mining begins. Unforeseen issues and market conditions, however, may lead to a lost of investment.

- Diversification is important to making a successful investment of risk capital. Never put all your eggs in one basket. The proportion of risk capital in a portfolio should be less than or equal to 10% and should offset with other more stable investments. Risk Capital should be the money that can afford to be lost.
**Application of Risk Capital**  Risk Capital is common in speculative investments and activities such as penny stocks, angel investing, lending, private equity, initial public offerings, real estate, day trading and swing trading of stocks, futures, options and commodities. Investors should evaluate the risk accurately before any investment. If the return is relative higher than the risk, then risk capital may be considered.

Day trading is another common use of Risk Capital, which is operated under the pattern day trading (PDT) rule. The PDT rule requires an account of at least $25,000. Accounts of more than $25,000 are allowed to make up to 4 roundtrip trades within a period of 5 days, while accounts of less than $25,000 are not allowed to do so. When an investor wants to short-sell stocks, margin is required.

### 2.2 Economic Capital

**Definition**  Economic capital is a measure of risk, which is the amount of capital that a firm, usually in financial services, needs to ensure that the company stays solvent given its risk profile. Economic capital is calculated internally, sometimes using proprietary models, and is the amount of capital that the firm should have to support any risks that it takes.

**Characteristics**

- To get the economic capital for a certain company, one should first convert a given risk into the amount of capital that is required to support it.

- The companys financial strength and the expected losses are the two keys of which the calculations of economic capital are based on.

- Financial strength is the probability of the firm not becoming insolvent over the measurement period and is the confidence level in the statistical calculation.

- Expected loss for a firm is the cost of doing business and is defined by the anticipated average loss divided by the period of measurement.

- The purpose of calculating the economic capital is to maximize the risk-reward trade-off for a certain company.
2.3 Regulatory Risk-Based Capital (RBC)

Definition  According to NAIC, Risk-Based Capital (RBC) is a method of measuring the minimum amount of capital appropriate for a reporting entity to support its overall business operations in consideration of its size and risk profile. This approach is a combination of deterministic projection by regulators and stochastic projections. In addition, the RBC calculation is confidential.

Characteristics  Risk-based capital is used to define a firm’s minimal capital it must hold in order to avoid regulatory intervention. The risk-based systems consist of RBC proportions. The sum of RBC charges equals an insurer’s risk-based capital. RBC limits the amount of risk a company can take. But RBC is not necessarily the full amount of capital that an insurer would like to hold to cover its possible loss, rather it is a minimum regulatory capital standard.

Due to different types of risks faced by the companies, the covariance formulas adopted for RBC ratios are different. The following formulae are developed to establish a minimum capital requirement that is able to cover the type of risk that the company is exposed to. Basic risk factors are asset risk, underwriting risk and other risk. For life insurance companies, RBC is based on the following assumptions:

\[
C_0 = \text{Asset Risk-Affiliates} \\
C_{1cs} = \text{Asset Risk-Unaffiliated Common Stock} \\
C_{1o} = \text{Asset Risk-Other Asset Risk} \\
C_2 = \text{Insurance Risk} \\
C_{3a} = \text{Interest Rate Risk} \\
C_{3b} = \text{Health Credit Risk} \\
C_{3c} = \text{Market Risk} \\
C_{4a} = \text{General Business Risk} \\
C_{4b} = \text{Administrative Expense Risk}
\]

\[
\text{Life Covariance} = C_0 + C_{4a} \sqrt{(C_{1o} + C_{3a})^2 + (C_{1cs} + C_{3c})^2 + (C_2)^2 + (C_{3b})^2 + (C_{4b})^2}.
\]  (2.1)
For the Property and Casualty companies, the RBC is calculated by

\[ R_0 = \text{Asset Risk-Subsidiary Ins. Companies} \]
\[ R_1 = \text{Asset Risk-Fixed Income} \]
\[ R_2 = \text{Asset Risk-Equity} \]
\[ R_3 = \text{Asset Risk-Credit} \]
\[ R_4 = \text{Underwriting Risk-Reserves} \]
\[ R_5 = \text{Underwriting Risk-Net Written Premium} \]

\[ P&C \text{ Covariance} = R_0 + \sqrt{(R_1)^2 + (R_2)^2 + (R_3)^2 + (R_4)^2 + (R_5)^2}. \] (2.2)

As to Health insurance companies, they assume

\[ H_0 = \text{Asset Risk-Affiliates} \]
\[ H_1 = \text{Asset Risk-Other} \]
\[ H_2 = \text{Underwriting Risk} \]
\[ H_3 = \text{Credit Risk} \]
\[ H_4 = \text{Business RBC} \]

\[ \text{Health Covariance} = H_0 + \sqrt{(H_1)^2 + (H_2)^2 + (H_3)^2 + (H_4)^2}. \] (2.3)

The RBC ratios are defined by

\[ \text{RBC Ratio} = \frac{\text{Total Adjusted Capital (TAC)}}{\text{Authorized Control Level RBC (ACL RBC)}}. \] (2.4)

where TAC is the actual amount of capital and surplus the company has and the summation of Statutory Capital and Surplus, Asset Valuation Reserve (AVR) including AVR in separate accounts, Half of company’s liability for dividends, company’s ownership share of AVR of subsidiaries, and Half of companies’ ownership share of subsidiaries’ dividend liability. In addition, ACL RBC (Hypothetical Minimum Capital) is 1 of 4 levels of calculated minimum capital.

The steps for the RBC calculation are as follows:

- Apply risk factors against annual statement values
- Sum risk amounts and adjust for statistical independence by applying (2.1), (2.2) and (2.3)
- Calculate ACL RBC
- Calculate RBC Ratio by (2.4)

Based on the RBC ratio, there are five action values in total for the companies or regulatory to perform. If the ratio is over 200%, there will be ‘no action’; if the ratio is 150% − 200%, it is the ‘company action level’; if the ratio is 100% − 150%, the insurer is required to make a plan and the state insurance commissions is required to issue orders or perform examinations to the insurer’s operations and business, which is the ‘regulatory action level’; if the ratio 70% − 100%, it turns to ‘authorized control level’; if the ratio is below ‘70%’, the regulator is required to take steps control the insurers and it is called ‘mandatory control level’.
3 Economic Profit Metrics, Ratios and Models

3.1 Metrics

- Economic value added (EVA)

\[(EVA)_j = (Net \ Income)_j - r_j C_j\]  \hspace{1cm} (3.1)

Economic value-added measures the return on an investment in excess of its expected or required return. \(r_j\) is the cost of capital for business \(j\). If \((EVA)_j \geq 0\), then continue writing the line of business; if \((EVA)_j < 0\), the line of business is reducing the firm’s market value.

\(EVA\) can be changed to a measure called economic value added on capital (EVAOC). It is defined as \(EVA\) divided by the capital allocated to a line.

\[(EVAOC)_j = \frac{(Net \ Income)_j}{C_j} - r_j\]  \hspace{1cm} (3.2)

This is similar to \(RAROC\). If \((EVAOC)_j \geq 0\), the line of business is creating value for the firm.

Even though these two approaches provide a good way of deciding whether a business line is creating value for the firm, they do not solve the problem of how to determine the cost of capital \(r_j\). To find cost of capital \(r_j\), we should introduce other capital allocation techniques.

- Value at Risk (VaR) is defined as the maximum amount the firm could lose over a specified time period with a specified probability. For example, in an insurance company, we usually calculate how much could be lost in on a calendar quarter at 1% probability level. To apply VaR into capital allocation, we usually use exceedance probabilities, which is defined as the probability that losses from a particular line of business will exceed the expected loss plus the capital allocated to the line. Capital can be allocated by equalizing the exceedance probabilities across the lines of business:

\[P[Loss_1 > E(Loss_1) + C_1] = \epsilon = P[Loss_2 > E(Loss_2) + C_2]\]  \hspace{1cm} (3.3)

3.2 Ratios

- Risk-adjusted return on capital (RAROC)

\[(RAROC)_j = \frac{(Net \ Income)_j - (Risk \ Adjustment \ to \ Net \ Income)_j}{C_j}\]  \hspace{1cm} (3.4)
The development of the RAROC method can be traced back to the late 1970s. It was initially used to measure the risk of the banks credit portfolio and the amount of equity capital necessary to limit the exposure of the banks depositors and other debt holders to a specified probability of loss. RAROC is defined as the net income from a line divided by the capital allocated to the line. It is used to evaluate the performance of business units for purposes of compensation line managers. $C_j$ is the capital allocated to the line of business $j$. Note that the net income in the formula should be after taxes and interest expense.

To decide whether the line of current risk-adjusted return is adequate, the risk-adjusted return should be compared with the cost of capital for business $j$. If the risk-adjusted return is greater than or equal to cost of capital, then continuing to devote resources to this line of business; if the risk-adjusted return is less than cost of capital, the line of business is reducing the firms market value.

- Return on Risk-Adjusted capital (RORAC) (Matten, 2000) The return on risk-adjusted capital (RORAC) is a rate of return statistic commonly used in financial analysis, where varying projects, endeavors and investments are evaluated based on capital at risk. Projects with different risk profiles are easier to compare to each other once their individual RORAC values have been calculated.

RORAC is based on estimated future earnings distributions or volatility of earnings. It is the firms capital specifically, which is adjusted for a maximum potential loss. Usually different corporate divisions use RORAC to quantify and maintain acceptable risk-exposure levels.

$$ (RORAC)_j = \frac{(Net \ Income)_j}{(Required \ Economic \ Capital)_j} \quad (3.5) $$

- Difference between RORAC and RAROC: The capital determined by RORAC is adjusted for risk and is used when risk varies. But the capital calculated by RAROC is adjusted for the rate of return.

- Risk-Adjusted Return on Risk-adjusted capital (RARORAC)
RARORAC is a combination of RAROC and RORAC, and proposes a measure considering the risk dimension both in the profitability of investments and economic capital allocated.

$$ (RARORAC)_j = \frac{(Net \ Income)_j - Risk \ Adjustment \ to \ Net \ Income)_j}{(Required \ Economic \ Capital)_j}. \quad (3.6) $$
In industry, the following formula is also adopted:

\[
(RARORAC) = \frac{|(r_p - r_f) - \beta_p (r_m - r_f)| CF_0}{Economic \ Capital},
\]

where \( r_p = \frac{CF_1}{CF_0} - 1 \) is the portfolio return with \( CF_0 \) and \( CF_1 \) being the initial investment at time \( t = 0 \) (utilized capital at risk) and the expected cash flow at time \( t = 1 \), \( r_f \) is the risk-free return, \( r_m \) is the market return, \( \beta_p \) is the systematic risk, \( r_p - r_f \) is the portfolio excess return. In the denominator, the economic/allocated/required capital is usually calculated by some risk measures, such as Value at Risk (VaR).

### 3.3 Models

**The Myers-Cohn (M-C) model** The M-C model determines the premium as the present value of the losses, expense, and tax cash flows expected to arise as a result of the coverage provided by the insurance contract. Cash flows are discounted at risk-adjusted discount rates, which are usually derived from CAPM.

\[
PV(P) = PV(L) + PV(E) + PV(TAX)
\]

where \( P = \) premium, \( L = \) losses, \( E = \) expenses, \( TAX = \) federal income taxes.

**The NCCI Model** The NCCI model concentrates on looking at the cash flows to and from the owners of the company. Different from the M-C model, it calculates an internal rate of return (IRR), which is compared to the cost of capital to decide whether the premium rates are adequate. Premium in this model is set at a level at which IRR equals cost of capital.

\[
0 = PV(\text{Equity Flows}) + PV(\text{Investment Income}) + PV(\text{Underwriting Profit})
\]

The premium is adjusted until the \( IRR = \) insurers cost of equity capital.

**The Insolvency Put Option Model** Consider a firm with assets \( A \) and liabilities \( L \) that is subject to default risk. If \( A > L \) at the liability maturity date, the policyholders receive the value of the liabilities and the owners or residual claimants receive the remaining assets of the firm; if \( A < L \), the owners of the firm default on the liability payment and the policyholders receive the assets of the firm.

To allocate capital, we need to determine the economic value of the policy holders claim.
on the firm prior to the liability maturity date, which should be the present value of the liabilities given the default risk is zero, minus a put option.

\[
\text{Value of Policyholder's Claim} = Le^{rt} - P(A, L, r, \tau, \sigma),
\]  

(3.10)

where \( P(A, L, r, \tau, \sigma) \) is the value of a solvency put option on \( A \) considering the expected policyholder deficit (EPD). Note that \( A \) is the value of asset, \( L \) is the value of liabilities, \( r \) is the risk-free rate, \( \tau \) is the time to maturity and \( \sigma \) is volatility. Capital assigned to each line is measured by the asset-to-liability ratio and assets are equal to liabilities plus the capital allocated to each line. So the asset-to-equity ratio equals \( \frac{1+C_j}{L_j} \).
4 Capital allocation vs asset allocation

In this chapter, we will compare capital allocation with asset allocation from various aspects to help individuals identify the capital allocation with asset allocation even though they both consider solvency or liquidity at certain level.

4.1 Asset Allocation

4.2 Definition

Asset allocation is the process of dividing investment among different kinds of assets, such as stock, bond, real estate and cash, to optimize the risk/reward trade-off based on an individuals or institutions specific situation and goals. Selecting the types of assets for a portfolio and allocating funds among different types of asset classes is the most significant step in asset allocation.

4.3 Types

There are two major sorts of asset allocation. According to Maginn (2010), strategic asset allocation is an integrative element of the planning step in portfolio management. In strategic asset allocation, an investors return objectives, risk tolerance, and investment constraints are integrated with long-run capital market expectation to establish exposure to IPS-permissible asset classes. The other type of asset allocation is tactical asset allocation. This involves making short-term adjustments to asset-class weight based on short-term expected relative performance among asset classes (Maginn et al. 2010).

According to Maginn (2010), in establishing a strategic asset allocation, an investment manager must specify a set of asset-class weights to produce a portfolio that satisfies the return and risk objectives and constraints as stated in the investment policy statement. One of critical step in strategic asset allocation is the Mean-Variance Approach. According to mean-variance theory, in determining a strategic asset allocation, an investor should choose from among the efficient portfolios consistent with investors risk tolerance. Efficient frontier graph is part of the minimum-variance frontier (MVF). Each portfolio on this frontier shows the portfolio with the smallest variance of return for its level of expected return. The turning point of this graph represents the global minimum-variance (GMV) portfolio (Maginn et al, 2010). The point has minimum standard deviation of return, which means lowest risk.
4.4 Capital Asset Pricing Model (CAPM)

To achieve asset allocation, it is important to obtain portfolio optimization. We can use Capital Asset Pricing Model (CAPM) to accomplish efficient portfolio allocation by balancing between the risk and the return. CAPM of William Sharpe (1964) and John Lintner (1965) marks the birth of asset pricing theory. Four decades later, CAPM is still widely used in applications, such as estimating the cost of equity capital for firms and evaluating the performance of managed portfolio (Eugene and Kenneth, 0000). CAPM predicts that the market portfolio consists of all available assets, each having a weight proportional to its market value, and the market risk factor, is the only factor capable to explain returns on asset (Falcao, Martelanc and Kazuo, 2016).

The CAPM model is

\[ \mathbb{E}[X_A] = X_f + \beta \left( \mathbb{E}[X_M] - X_f \right) \]  \hspace{1cm} (4.1)

- \( X_A \) = the rate of return
- \( \mathbb{E}[X_A] \) = the expected return on the account given its \( \beta \)
- \( X_f \) = the risk-free rate of return
- \( X_M \) = the market rate of return
- \( \mathbb{E}[X_M] \) = the expected return on the market portfolio
- \( \beta_A \) = the account’s \( \beta \) or sensitivity to returns on the market portfolio

\[ \beta_A = \frac{Cov[X_A, X_M]}{Var[X_M]} \]

CAPM can let investors know which stock is riskier. Even though it is not easy to predict from beta how each stock moves, investors can eliminate the high beta stock when they create portfolio. For risk-averse investors, CAPM is one of efficient models to allocate asset.

4.5 Black-Litterman Model

Black-Litterman Model is another part of an asset allocation process. The Black-Litterman Model was first published by Fischer Black and Robert Litterman in 1990. Black-Litterman model makes two significant contributions to the problem of asset allocation (Walters, 2007). The Black-Litterman model provides for estimates, which lead to more stable and more diversified portfolios than estimates derived from historical returns when used with unconstrained mean-variance optimization (Walters, 2007). The steps for allocating assets are as follows:
• Determine which assets constitute the market
• Compute the historical covariance matrix for the assets
• Determine the market capitalization for each asset class
• Use reverse optimization to compute the CAPM equilibrium returns for the assets
• Specify views on the market
• Blend the CAPM equilibrium returns with the views using the Black-Litterman model
• Feed the estimates (estimated returns, covariances) generated by the Black-Litterman model into a portfolio optimizer
• Select the efficient portfolio which matches the investors risk preferences

The Black-Litterman model is

$$
\mathbb{E}[R] = \left[ (\tau \Sigma)^{-1} + P^T \Omega^{-1} P \right]^{-1} \left[ (\tau \Sigma)^{-1} \Pi + P^T \Omega^{-1} Q \right],
$$

where $\mathbb{E}[R]$ is the new (posterior) Combined Return Vector ($N \times 1$ column vector), $\tau$ is a scalar, $\Sigma$ is the covariance matrix of excess return ($N \times N$ matrix), $P$ is a matrix that identifies the assets involved in the views ($K \times N$ matrix or $1 \times N$ row vector in the special case of 1 view), $\Omega$ is a diagonal covariance matrix of error terms from the expressed views representing the uncertainty in each view ($K \times K$ matrix), $\Pi$ is the Implied Equilibrium Return Vector ($N \times 1$ column vector), and $Q$ is the View Vector ($K \times 1$ column vector).

4.6 Relationship with capital allocation (Similarity and Difference between asset allocation and capital allocation)

4.6.1 Similarity

Both asset allocation and capital allocation have trade-off between risk and expected rate of return. One is to maximize the expected rate of return of the company and the other is to minimize the risk of the firm.

4.6.2 Differences

According to Spaulding (2017), asset allocation is the assortment of funds among different types of assets with different ranges of expected returns and risk. Bonds, real estate, stocks
and foreign assets are included in asset. Capital allocation, on the other hand, is the division of funds between risk-free investments, such as T-bills, and risky assets such as stocks.

When allocating capital, it is important to demonstrate how different degrees of risk aversion will affect allocations between risky and risk-free assets. We can say that asset allocation is one example of capital allocation. The riskier bonds usually have higher average return. It means that there always exist trade-off between the risk and expected return. In this situation, the investors do not make all-or-nothing choices from investment classes. They can construct portfolios using securities from all asset classes, which means they can combine their portfolio with risky asset and risk-free asset considering the Capital Allocation Line (CAL).

Capital Allocation Line (CAL) measures the risk of risky asset and risk-free asset. CAL is

$$E[r_c] = r_f + \frac{\sigma_c}{\sigma_p}(E[r_p] - r_f)$$

(4.3)

- $r_f$ = the rate of return on the risk-free asset
- $r_p$ = the rate of return on the risky portfolio
- $r_c$ = the rate of return on the complete portfolio
- $\sigma_p$ = the standard deviation of the return on the risky portfolio
- $\sigma_c$ = the standard deviation of the return on the complete portfolio

If $\frac{\sigma_c}{\sigma_p}$ is higher than 1, there is a negative investment in the risk-free asset. It means that the investor borrowed at the risk-free rate. The complete portfolio has higher expected return, but higher variance (risk). If the investors are high-level risk-aversion, it leads to larger proportion of investment in the risk-free asset. However, if the investors are low-level risk aversion, it leads to larger proportion of investment in the portfolio of risky assets. If investors accept high-level of risk for high-level of return, it will result in leveraged investment.
5 Policy limits and deductibles allocation principles

In the literature, capital allocation also includes the allocation of policy limits and deductibles (Cheung, 2007) as well as the proportional capital allocations in the introduction section in this paper. Later on, scholars built on Cheung’s model and extend the results to more general cases. Firstly, we will provide the definitions of stochastic orders since Cheung’s results are most based on order of the allocated policy limits or deductibles under different model assumptions.

5.1 Preliminaries

5.1.1 Preliminary: Stochastic Orders

Let $X$ and $Y$ be two random variables, $f(\cdot)$ and $g(\cdot)$ be their probability density/mass functions, $F(\cdot)$ and $G(\cdot)$ be their cumulative distribution functions, and $\bar{F}(\cdot)$ and $\bar{G}(\cdot)$ be their survival functions.

Usual Stochastic Order

$$X \preceq_{st} Y \iff F(t) \leq \bar{G}(t) \text{ for } \forall t \iff \mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)] \text{ for } \forall \text{ increasing function } \phi$$

(5.1)

Increasing Convex Order

$$X \preceq_{icx} Y \iff \mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)] \text{ for } \forall \text{ increasing convex function } \phi$$

(5.2)

Increasing Concave Order

$$X \preceq_{icv} Y \iff \mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)] \text{ for } \forall \text{ increasing concave function } \phi$$

(5.3)

Hazard Rate Order

$$X \preceq_{hr} Y \iff \frac{\bar{G}(t)}{F(t)} \text{ is increasing in } t$$

(5.4)

Reversed Hazard Rate Order

$$X \preceq_{rh} Y \iff \frac{G(t)}{F(t)} \text{ is increasing in } t$$

(5.5)
Likelihood Ratio Order

\[ X \leq_{lr} Y \iff \frac{g(t)}{f(t)} \text{ is increasing in } t \] (5.6)

Relationship

\[
\begin{align*}
X \leq_{lr} Y & \Rightarrow X \leq_{hr} Y \\
X \leq_{lr} Y & \Rightarrow X \leq_{rh} Y \\
X \leq_{hr} Y & \Rightarrow X \leq_{st} Y \\
X \leq_{rh} Y & \Rightarrow X \leq_{st} Y \\
X \leq_{st} Y & \Rightarrow X \leq_{icx} Y \\
X \leq_{st} Y & \Rightarrow X \leq_{icv} Y
\end{align*}
\]

5.2 Allocation of Policy Limits and Deductibles

5.2.1 Cheung’s Classic Model

Suppose \( w \) is the given current wealth level, \( X_j, j = 1, \ldots, n \) are independent losses for each business line, and \( l_j \) and \( d_j \) are the corresponding policy limit and deductible for the \( j \)th business line. Cheung (2007) studied the problem by focusing on total retained loss, which can be expressed in the following forms under policy limits and deductibles respectively.

\[
\begin{align*}
\sum_{j=1}^{n} X_j - \sum_{j=1}^{n} (X_j \wedge l_j) = \sum_{j=1}^{n} (X_j - l_j)_+ & \quad (5.7) \\
\sum_{j=1}^{n} X_j - \sum_{j=1}^{n} (X_j - d_j)_+ = \sum_{j=1}^{n} (X_j \wedge d_j) & \quad (5.8)
\end{align*}
\]

He further considered three different loss functions: expected utility of wealth with known dependent structure, expected retained loss, and expected utility of wealth with unknown dependent structure. Thus, there are six optimization problems to be solved.

Policy Limit: Maximize Expected Utility of Wealth with Known Dependent Structure

The objective function is

\[
\max_{(l_1, \ldots, l_n)} E[u(W - \sum_{j=1}^{n} (X_j - l_j)_+)] \quad \text{with } u(\cdot) \text{ is increasing and concave}
\]

and the conclusion for the order of the allocated limits is that

\[ X_j \leq_{hr} X_j \Rightarrow l_j^* \leq l_j^* \]
Policy Limit: Minimize Expected Total Retained Loss The objective function is
\[
\max_{(l_1,\ldots,l_n)} E\left[\sum_{j=1}^{n} u((X_j - l_j)_+)\right] \text{ with } u(\cdot) \text{ is increasing and concave}
\]
and the conclusion for the order of the allocated limits is
\[
X_j \leq_{st} X_j \Rightarrow l_j^* \leq l_j^*.
\]

Policy Limit: Maximize Expected Utility of Wealth with Unknown Dependent Structure
The objective function is
\[
\max_{(l_1,\ldots,l_n)} E[u(W - \sum_{j=1}^{n} (X_j - l_j)_+)] \text{ with } u(\cdot) \text{ is increasing and concave}
\]
and the conclusion for the order of the allocated limits is
\[
X_j \leq_{st} X_j \Rightarrow l_j^* \leq l_j^*.
\]

Deductible: Maximize Expected Utility of Wealth with Known Dependent Structure
The objective function is
\[
\max_{(d_1,\ldots,d_n)} E[u(W - \sum_{j=1}^{n} (X_j \wedge d_j))] \text{ with } u(\cdot) \text{ is increasing and concave}
\]
and the conclusion for the order of the allocated deductible is
\[
X_j \leq_{hr} X_j \Rightarrow d_j^* \geq d_j^*.
\]

Deductible: Minimize Expected Total Retained Loss The objective function is
\[
\max_{(d_1,\ldots,d_n)} E[u\left(\sum_{j=1}^{n} (X_j \wedge d_j)\right)] \text{ with } u(\cdot) \text{ is increasing and concave}
\]
and the conclusion for the order of the allocated deductible is
\[
X_j \leq_{st} X_j \Rightarrow d_j^* \geq d_j^*.
\]

Deductible: Maximize Expected Utility of Wealth with Unknown Dependent Structure
The objective function is
\[
\max_{(d_1,\ldots,d_n)} E[u(W - \sum_{j=1}^{n} (X_j \wedge d_j))] \text{ with } u(\cdot) \text{ is increasing and concave}
\]
and the conclusion for the order of the allocated deductible is
\[
X_j \leq_{st} X_j \Rightarrow d_j^* \geq d_j^*.
\]
**Conclusion and Conjecture**  These results are intuitive. As Cheung concludes, ‘if the potential loss is relatively large, it is optimal to allocate larger policy limits and smaller deductible.’ He also conjectured that the relationship should hold for variability, not just limited to size. Zhuang et al. (2009) extended Cheung’s model to the case under distortion risk measure, and considered frequency impacts while Cheung’s model only studied severity.
6 Capital Allocation Principles

6.1 Capital Allocation Models

Suppose there are \( n \) business lines, and the potential loss for the \( j \)th business line is \( X_j \), \( j = 1, \ldots, n \), and the total capital budget is \( C \), which will be allocated to each business line with amount \( C_j \).

6.1.1 Unified Capital Allocation Model (Dhaene et al., 2012)

In Dhaene et al. (2012), capital allocation principles are determined by the optimizers \((C_1, \ldots, C_n)\) for the unified optimization problem:

\[
\min_{(C_1, \ldots, C_n) \in \mathbb{R}^n} \sum_{j=1}^{n} \nu_j \mathbb{E}[\xi_j D(\frac{X_j - C_j}{\nu_j})], \quad \text{s.t.} \quad \sum_{j=1}^{n} C_j = C,
\]

where \( \nu_j, j = 1, \ldots, n \), are non-negative real numbers such that \( \sum_{j=1}^{n} \nu_j = 1 \), \( \xi_j, j = 1, \ldots, n \), are non-negative with \( \mathbb{E}[\xi_j] = 1 \), and \( D \) is a non-negative function. Most of the proportional capital allocation principles, such as the principles based on distortion risk measures, as well as the sharp ratio theories are unified by the model.

6.1.2 Capital Allocation Problems Based on Weighted Loss Functions (Cai and Wang, 2016)

From a global perspective, Cai and Wang (2016) introduced a general model minimizes the total capital deficit and surplus risk at the same time based on Dhaene et al. (2010). They introduced Capital Allocation Problem II:

\[
\min_{(C_1, \ldots, C_n) \in \mathbb{R}^n} \sum_{j=1}^{n} \nu_j \mathbb{E}[\xi_j D_1(\frac{X_j - C_j}{\nu_j})] + \omega_j \mathbb{E}[\psi_j D_2(\frac{X_j - C_j}{\omega_j})], \quad \text{s.t.} \quad \sum_{j=1}^{n} C_j = C, \tag{6.1}
\]

where \( \nu_j \) and \( \omega_j \), \( j = 1, \ldots, n \), are non-negative constants; \( \xi_j \) and \( \psi_j \) are functionals of \( X_j \), \( j = 1, 2, \ldots, n \).

In this model, \((X_j - C_j)_-\) represents the capital obtained by the \( j \)th business line over potential loss, and is defined as the ‘capital surplus risk’ for the \( j \)th business line. Additionally, \((X_j - C_j)_+\) is defined as the ‘capital deficit risk’ for the \( j \)th business line.

**Theorem 6.1.** Let \( D_1, D_2 : \mathbb{R}_+ \to \mathbb{R}_+ \) be non-degenerated, convex and increasing functions. For Capital Allocation Problem II, there exists at least one global minimizer, and \( C_j, j = 1, \ldots, n \)
1, 2, . . . , n, the global optimal allocated capital on the jth business line, and λ, an auxiliary real-valued variable, are the solutions to

\[
\begin{align*}
\mathbb{E}[\psi_j D'_{2+}(X_j - C_j) - \frac{\nu_j}{\omega_j}] &\leq \mathbb{E}[\xi_j D'_{1-}(X_j - C_j) - \frac{\nu_j}{\omega_j}] - \lambda, \ j = 1, 2, \ldots, n, \\
\mathbb{E}[\xi_j D'_{1+}(X_j - C_j) - \frac{\nu_j}{\omega_j}] &\leq \mathbb{E}[\psi_j D'_{2-}(X_j - C_j) - \frac{\nu_j}{\omega_j}] + \lambda, \ j = 1, 2, \ldots, n, \\
\sum_{j=1}^{n} C_j &= C.
\end{align*}
\]

If \( D_1 \) and \( D_2 \) are differentiable with \( D'_1(0) = D'_2(0) = 0 \), then \( C_j, \ j = 1, 2, \ldots, n \) and \( \lambda \), an auxiliary real-valued variable are the solutions to

\[
\begin{align*}
\mathbb{E}[\psi_j D'_{2}(X_j - C_j) - \frac{\nu_j}{\omega_j}] &= \mathbb{E}[\xi_j D'_{1}(X_j - C_j) - \frac{\nu_j}{\omega_j}] - \lambda, \ j = 1, 2, \ldots, n, \\
\sum_{j=1}^{n} C_j &= C.
\end{align*}
\]

Based on Theorem 6.1, for the quadratic case, they defined the ‘add on and off capital allocation principle’ by

\[
C_j = \frac{\sum_{j=1}^{n} \nu_j}{\sum_{j=1}^{n} \frac{\nu_j}{\omega_j}} K + \frac{\sum_{j=1}^{n} \nu_j}{\sum_{j=1}^{n} \frac{\nu_j}{\omega_j}} \left( \sum_{j=1}^{n} \frac{\nu_j}{\omega_j} \mathbb{E}[\xi_j X_j] - \mathbb{E}[\xi_j X_j] \right), \quad (6.2)
\]

When \( \frac{\xi_j}{\nu_j} = \frac{\psi_j}{\omega_j} \) and \( D_1(x) = D_2(x) = x^2 \), \((6.1)\) is reduced

\[
\min_{(C_1, \ldots, C_n) \in \mathbb{R}^n} \sum_{j=1}^{n} \nu_j \mathbb{E}[\xi_j (X_j - C_j)^2], \text{ s.t. } \sum_{j=1}^{n} C_j = C.
\]

Moreover, if \( \sum_{j=1}^{n} \nu_j = 1 \) and \( \mathbb{E}[\xi_j] = 1 \), Theorem 6.1 can be reduced as in Theorem 1 of Dhaene et al. (2012). Also, the aggregate portfolio driven allocation principles in Table 2 of Dhaene et al. (2012) can be included in this revised model.

### 6.2 Capital allocation principles with sub-business lines

For the jth business line, there are \( n_j, \ j = 1, \ldots, n \) sub-business lines. The capital allocated to the i-the sub-business line for the j-the business line to mitigate the corresponding loss \( X_{i,j} \) is \( C_{i,j} \). Following the same idea in Cai and Wang (2016), the model is revised to

\[
\min_{(C_{i,j}, j = 1, \ldots, n_j, i = 1, \ldots, n)} \sum_{j=1}^{n} \sum_{i=1}^{n_j} \left\{ \nu_{i,j} \mathbb{E}[\xi_{i,j} D_1(\frac{X_{i,j} - C_{i,j}}{\nu_{i,j}})] + \omega_{i,j} \mathbb{E}[\psi_{i,j} D_2(\frac{X_{i,j} - C_{i,j}}{\omega_{i,j}})] \right\},
\]

\[(6.3)\]
such that
\[
\sum_{j=1}^{n_j} C_{i,j} = C_j, \ j = 1, \ldots, n
\]
and
\[
\sum_{j=1}^{n} \sum_{j=1}^{n_j} C_{i,j} = C.
\]

**Theorem 6.2.** Let \(D_1, D_2 : \mathbb{R}_+ \to \mathbb{R}_+\) be non-degenerated, convex and increasing functions. For Capital Allocation Problem II, there exists at least one global minimizer, and \(C_{i,j}, i = 1, \ldots, n, j = 1, 2, \ldots, n\), the global optimal allocated capital on the \(i\)the sub-business line for the \(j\)the business line, and \(\lambda\), an auxiliary real-valued variable, are the solutions to

\[
\begin{cases}
\mathbb{E}[\psi_{i,j} D_2' \left( \frac{(X_{i,j} - C_{i,j}) - \omega_{i,j}}{\nu_{i,j}} \right) \mathbb{1}_{\{X_{i,j} \leq C_{i,j}\}}] \geq \mathbb{E}[\xi_{i,j} D_1' \left( \frac{(X_{i,j} - C_{i,j}) + \omega_{i,j}}{\nu_{i,j}} \right) \mathbb{1}_{\{X_{i,j} > C_{i,j}\}}] - \lambda, \\
\mathbb{E}[\xi_{i,j} D_1' \left( \frac{(X_{i,j} - C_{i,j}) + \omega_{i,j}}{\nu_{i,j}} \right) \mathbb{1}_{\{X_{i,j} \geq C_{i,j}\}}] \geq \mathbb{E}[\psi_{i,j} D_2' \left( \frac{(X_{i,j} - C_{i,j}) - \omega_{i,j}}{\nu_{i,j}} \right) \mathbb{1}_{\{X_{i,j} < C_{i,j}\}}] + \lambda, \\
\sum_{j=1}^{n_j} C_{i,j} = C_j, \ j = 1, \ldots, n, \\
\sum_{j=1}^{n} C_j = C.
\end{cases}
\]

If \(D_1\) and \(D_2\) are differentiable with \(D_1'(0) = D_2'(0) = 0\), then \(C_{i,j}, j = 1, \ldots, n_j, j = 1, 2, \ldots, n\) and \(\lambda\), an auxiliary real-valued variable are the solutions to

\[
\begin{cases}
\mathbb{E}[\psi_{i,j} D_2' \left( \frac{(X_{i,j} - C_{i,j}) - \omega_{i,j}}{\nu_{i,j}} \right)] = \mathbb{E}[\xi_{i,j} D_1' \left( \frac{(X_{i,j} - C_{i,j}) + \omega_{i,j}}{\nu_{i,j}} \right)] - \lambda, \ i = 1, \ldots, n_j, \ j = 1, 2, \ldots, n, \\
\sum_{j=1}^{n_j} C_{i,j} = C_j, \ j = 1, \ldots, n, \\
\sum_{j=1}^{n} C_j = C.
\end{cases}
\]

### 6.3 Quadratic Capital Allocations Principles with Sub-business Lines

If we suppose both \(D_1(x)\) and \(D_2(x)\) are quadratic functions, we can arrive at the following theorem; if both \(D_1(x)\) and \(D_2(x)\) are identical functions, similar results can be obtained as
in Cai and Wang (2016), but the principles are more complex than the quadratic case, and especially, the capital allocation might not be unique. Thus, in this section we only provide the conclusion for the quadratic case.

**Theorem 6.3. (Quadratic Capital Allocation Principles)** If \( D_1(x) = D_2(x) = x^2 \), then \( C_{i,j}, j = 1, \ldots, n_j, j = 1, 2, \ldots, n \), the global optimal allocated capital on the \( i \)th sub-business line for the \( j \)th business line, and \( \lambda \), an auxiliary real-valued variable, are the solutions to

\[
\begin{align*}
\mathbb{E}\left[ \frac{\xi_{i,j}}{\nu_{i,j}} (X_{i,j} - C_{i,j})_+ \right] &= \mathbb{E}\left[ \frac{\psi_{i,j}}{\omega_{i,j}} (X_{i,j} - C_{i,j})_- \right] + \frac{\lambda}{2}, \quad i = 1, \ldots, n_j, \quad j = 1, 2, \ldots, n, \\
\sum_{j=1}^{n_j} C_{i,j} &= C_j, \quad j = 1, \ldots, n, \\
\sum_{j=1}^{n} C_j &= C.
\end{align*}
\]
References


25


