Reinsurance Framework
-- Regulations, Products and Strategies

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Chapter 1

Introduction

Reinsurance is usually adopted to mitigate catastrophic type loss. Nowadays, both the external environment and internal situation change instantly from the perspective of either a country or a family. It might be safe for insurance companies to enlarge the products' volume since the risk can be diversified according to the law of large numbers. However, the characteristic of risk centralization in recent years, such as the financial crisis and natural disaster, may lead the insurance companies to bankrupt if they do not take other risk management strategies to mitigate them. Reinsurance is a method companies can use to re-diversify the risk centralization. It is because reinsurance companies can diversify the risks geographically according to the categories of risk factors. Also, reinsurance companies can provide consulting services to their clients, help finance new business and satisfy other needs.

In this paper, we are going to illustrate a framework for reinsurance, the regulations across various countries, the reinsurance products issued by the leading reinsurance companies and the optimal reinsurance strategies, or the optimal ceded functions that used to transfer loss from the insurance companies to the reinsurance companies from academic papers.
Chapter 2

History and Practice of Reinsurance

2.1 History of Reinsurance

Reinsurance is a device used by insurance companies to avoid catastrophic losses. Since reinsurance is a form of insurance, its history is entwined with the development of insurance. Ways for spreading risks could be traced back to earliest days of commercial transactions. For instance, ancient Chinese merchants would separate their goods among multiple vessels for going through hazardous rivers of China. In 3000 BC, the Babylonians formulated ways of marine loans which helped the borrower avoid reimbursing either the loan or the interest if certain mishaps happened. The present-day form of insurance originated in northern Italy then flourished in England. In 1310, the Count of Flanders agreed a charter for the foundation of a Chamber of Assurance to deal with underwriting risk of marine journey. The Great Fire of London in 1666 stimulated the establishment of numerous fire insurance companies. The increasing growth of insurance industry came a demand for reinsurance.

The first policy of a marine reinsurance was written by Gustav Cruciger to address the issue in 1370 for a voyage from Genoa to Sluys. The original insurer then reinsured the riskiest section of the route from Cadiz to Sluys. He utilized reinsurance to keep away from a hazardous risk which he favored not to carry but had been obliged to accept in order to gain the more desirable business and keep a profitable client. The origins of life reinsurance occurred during the first half of the 19th century. It happened very late because there were various challenges and disputes of reinsurance. For example, original insurers did not always disclose retentions and reinsurances were not always cancelled following the discontinuance of the original policy. In 1918, a company in England called the Mercantile and General started the life reinsurance business. However, the life reinsurance treaties did not appear until 1927 due to the introduction of risk premium rates. The earliest accident reinsurance occurred in 1872. The Railway Passengers Assurance Company set an agreement with a life assurance company to pay the liability for accidental death claims exceeding 2000 pounds in total in one vessel for immigrants going to New Zealand.

In general, reinsurance is that an insurer transfers part of its risk to another insurer by paying a reinsurance premium. There are some other ways of non-traditional insurance to deal with risks, such as coinsurance, self-insurance and captive insurance. Coinsurance is that two or more insurers come together to share the risk in certain proportion. Compared with reinsurance,
coinsurance has a drawback that it introduces its clients to its competitors. Self-insurance is that a company reserves part of money to insure risks by itself without the involvement of third party. The most common example of self-insurance is that an employer offers health insurance to its employees from its own assets. However, the limitation of self-insurance is that a company must be certain that its self-insured risk is under the control. Captive insurer is an insurance company entirely owned by its insureds, so its primary aim is to insure the risk of its owners. In fact, any insured who purchases captive insurance also puts its own capital at risk since captive insurance is not simply an insurance but also an investment.

There are two basic methods for arranging different types of reinsurances: facultative and treaty. In the early days, reinsurances were placed facultatively. Each risk was offered individually to another insurer, who was free to accept or reject the offer. It was not until the middle of the 19th century that the first reinsurance treaty appeared. Treaty emphasizes that there is an obligation on the reinsurer to accept risks within the terms and conditions of the agreement.

2.2 Types of Reinsurance

Reinsurance can be classified into various ways. It can be classified based on the ceding mechanism, which results in a facultative or treaty contract, and how the risk is shared, which results in a proportional or non-proportional contract.

2.2.1 Facultative vs. Treaty

A facultative reinsurance involves the reinsurer underwriting each risk separately. Normally, it is used by a direct insurer who does not have the capacity to write a certain risk by itself. For example, a direct insurer goes to the reinsurer to help write insurance for a property risk which it cannot handle on its own due to the high total insurable value. When writing a facultative reinsurance agreement, reinsurer has the right to accept or reject each risk and profit is expected more quickly depending on the reinsurer's risk selection process. Moreover, insurers can also choose which risk to cede to the reinsurer, allowing flexibility.

On the other hand, a treaty reinsurance contract involves a pre-negotiated agreement between the direct insurer and the reinsurer, in which underwriting is executed during the negotiation of the contract. Generally, treaty is renewed every year. The primary insurer agrees to cede all risks within a defined class or classes to the reinsurer, and the reinsurer agrees to reinsure all the risks without the need of individual risk underwriting. Hence, the risk is bundled within a portfolio rather than separate. Due to the nature of a treaty reinsurance contract, reinsurers normally expect a long-term relationship with the insurer and profitability is adjusted over an extended period. Also, treaty reinsurance contracts cost less than facultative reinsurance contracts.
2.2.2 Proportional vs. Non-Proportional

For the *proportional* reinsurances, there are mainly two types: *quota-share and surplus-loss*. Quota-share will utilize a constant proportion to determine the ceded loss and the reinsurance premium, while the proportion in surplus-loss is determined by the amount of claims and might not be a fixed percentage.

*Non-proportional* also have two types, *stop-loss* and *excess-of-loss*. The reinsurer is only liable to pay if the losses incurred by the ceding company exceed some predetermined amount. Stop-loss is a type of reinsurance managing aggregate loss ratio. For example, if the total amount of benefit claims equal 120% of the face amount of one policy, this insurance would cover the additional 20% of the total costs.

Excess-of loss contract indemnifies a ceding company, up to a certain limit, against the amount of loss in excess of a specified retention. For example, assume a company buys a 500,000 xs 500,000 excess-of-loss reinsurance contract. It means the company will retain the first $500,000 of losses and the reinsurance company will cover losses above that $500,000.

2.3 Reinsurance Practice

Facultative reinsurance usually is used for properties that have two conditions. Insured value is *too high and too rare* to generalize such as special case of marine, fire, airplane and catastrophe (CAT). Pricing and contract conditions are almost decided by veteran underwriter.

Treaty reinsurance is available for most insurable risks. Although insurance companies utilize various types of treaty, they can be generalized as follows.

- Proportional reinsurance: Quota-Share is a basic type of reinsurance. This type of reinsurance is very useful for risk that has low severity and high frequency such as health and life insurance. Insurer who wants to take risk but the insured amount is too large to take alone will utilize the surplus share.

- Non-proportional reinsurance: If loss ratio persists at a certain level, neither too low nor too high, stop-loss would be the best way. Because it is managed as an aggregation of losses, liabilities, properties and some kinds of health insurance are common to use stop-loss. Excess of loss is the closest reinsurance type to the intent of the insurance itself. Therefore, XOL is adopted for properties that have a probability of huge loss, such as fire, marine, airplane and CAT.
Chapter 3

Reinsurance Regulations

In this chapter, we will investigate the regulations on reinsurance practice across different countries. In general, insurance and reinsurance companies are required to obtain a minimum capital or surplus to pay for the potential loss, and to be licensed by a government or national insurance Authority, Commission or Regulations, which is illustrated in Table 3. In addition, some countries, such as Japan, require the insurance companies to transfer risk to foreign reinsurance companies, however, some government regulate that the foreign reinsurers are not supposed to hold more than certain amount of risk.

<table>
<thead>
<tr>
<th>Country</th>
<th>Reinsurance Regulator(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>US</td>
<td>National Association of Insurance Commissions (NAIC)</td>
</tr>
<tr>
<td>Canada</td>
<td>Office of the Superintendent of Financial Institutions (OSFI)</td>
</tr>
<tr>
<td>Mexico</td>
<td>National Insurance and Bonding Commission (CNSF)</td>
</tr>
</tbody>
</table>
| UK      | Prudential Regulation Authority (PRA)  
          | Financial Conduct Authority (FCA) |
| Switzerland | Swiss Financial Market Supervisory Authority (FINMA)  
               | Federal Insurance Supervision Act (ISA) |
| Denmark | Danish Financial Supervisory Authority (DFSA) |
| Netherlands | De Nederlandsche Bank (DNB)  
             | Financial Supervision Act (FSA) |
| Australia | Australian Prudential Regulation Authority (APRA) |
| China   | China Insurance Regulatory Commission (CIRC) |
| Malaysia | Malaysia Insurance Act (MIA) |
| Japan   | Insurance Business Act (IBA) |
| India   | Insurance Regulatory and Development Authority Act, 1999  
          | Insurance Regulatory and Development Authority  
          | Insurance Advisory Committee |

Table 3
3.1 North America

3.1.1 United States

In the United States, reinsurers are normally regulated on a state level. Each state runs autonomously to supervise their insurance markets, normally through a state department of insurance or division of insurance. Regulation by states is under two main different methods: direct regulation of US licensed reinsurers and indirect regulation of reinsurance transactions.

In regulating reinsurers, states focus mainly on the reinsurer's solvency. Some requirements related to solvency include minimum capital and surplus requirements, risk-based capital requirements, investment restrictions, asset valuation requirements, actuarial-certified loss reserve opinion statements, etc.

One important document "US reinsurance collateral white paper" is proposed by National Association of Insurance Commissioners (NAIC), the standard-setting and regulatory support organization created and governed by the chief insurance regulators from the 50 states. In that document, it particularly points out that a U.S. ceding company will not be permitted to take statutory credit or paid loss amounts recoverable from reinsurers as an asset on its balance sheet, unless such reinsurers meet one of the following requirements: 1) The reinsurer is licensed in the same state of domicile as the ceding company for a like kind of business. 2) The domiciliary insurance department of the ceding company accredits the reinsurer. 3) The reinsurer is domiciled and licensed in a state with substantially similar credit for reinsurance laws as the state of the ceding company. 4) The reinsurer provides collateral in the form of a multiple beneficiary trust. 5) The reinsurer provides collateral or other security to the ceding insurer.

Regarding indirect regulation, regulators want to make sure that reinsurance will be fully payable. There are several criteria. One of these criteria involves a collateral requirement where non-licensed reinsurers must either establish a US trust fund or have another security in the US that will unconditionally cover the potential liabilities of the insurer (such as a bank).

3.1.2 Canada

Most of the world's reinsurance capacity is provided by a relatively small number of large global reinsurance enterprises operating out of select countries. The reinsurance industry in Canada reflects this trend and comprises most foreign-based enterprises.

In Canada, on the other hand, there is an independent agency of the Government of Canada called Office of the Superintendent of Financial Institutions (OSFI). It is the sole regulator of
banks, and the primary regulator of insurance companies, trust companies, loan companies and pension plans.

There are statutory limits and requirements related to transferring risk by insurance and reinsurance companies. In ordinary course, insurers and reinsurers need to have a board-approved and monitored reinsurance risk management policy designed to ensure prudential placement and management of all reinsurance risks. Insurers that cede risks to unlicensed reinsurers (affiliated or not) are not permitted to take capital credit for the reinsurance unless certain conditions are fulfilled, such as: 1) There must be a qualifying collateral arrangement with the reinsurer (called a reinsurance security agreement). 2) For reinsurance arrangements such as those underwritten on a "funds withheld" basis, the reinsurance transaction must contractually satisfy the Superintendent of Financial Institutions' requirements. For example, the reserves (comprised of premiums withheld) must legally form part of the estate of the ceding company should an insolvency occur.

Also, there are main exceptions or exclusions from authorization or licensing for reinsurance. For example, reinsurers are not required to be licensed in certain western provinces as long as their business in the province is confined or restricted to reinsurance, and generally not required to be licensed provincially if the reinsurance is transacted "outside" the province.

3.2 Central America

3.2.1 Mexico

In Mexico, the Insurance Contract Law (LCS) implemented since 1935 and the Insurance and Surety Companies Law (LISF) in effect since April 2015 primarily govern the business of reinsurance companies. In addition, the National Insurance and Bonding Commission (CNSF), as an independent agency of the Ministry of Finance, has major responsibility for regulating and supervising reinsurance operations in Mexico. CNSF is responsible for supervising the operation of the insurance and surety industries to guarantee the compliance with the regulatory framework and the maintenance of institutions solvency and financial stability.

In particular, the insolvency of Mexican reinsurance companies is regulated by the special provisions in Title Twelve of the Insurance Law and Surety Companies Law (LISF). It has the following requirements: 1) In the conciliation stage, the parties negotiate an agreement for the company to continue operating. This stage cannot last for more than 365 calendar days. 2) The bankruptcy stage starts when the parties do not reach a non-bankruptcy agreement, and its objective is to sell the insolvent insurance company assets to pay its creditors.

There are also statutory limits and requirements relating to the transfer of risk by reinsurance companies. Generally, insurance and reinsurance companies can only transfer risk of their
insurance portfolio by one of the following: 1) Ceding risks in reinsurance to a Mexican insurance or reinsurance company, or a foreign reinsurance company registered with the Reinsurance Registry. 2) Transferring the insurance portfolio, in the terms set out in Article 270 of the Insurance Law and Surety Companies Law (LISF), in which case prior authorization from the National Insurance and Bonding Commission (CNSF) is required.

3.3 Europe

3.3.1 United Kingdom (UK)

In the UK, the Financial Services and Markets Act 2000 as amended, and the Financial Services and Markets Act 2000 (Regulated Activities) Order 2001 provide the framework for the regulation of insurance and reinsurance activities. The Financial Conduct Authority's Handbook and the Prudential Regulation Authority's Handbook provide detailed rules and guidance on governance, capital and conduct of business requirements. Therefore, the regulation of insurers and reinsurers is undertaken by the Prudential Regulation Authority (PRA) for prudential purposes, and Financial Conduct Authority (FCA) for conduct purposes.

If a firm plans to operate reinsurance business, it must obtain the Part IVA FSMA permission from the Prudential Regulation Authority (PRA). Exception may apply when the firm is exempt or able to rely on the EU's passporting regime. In order to get a valid Part IVA FSMA permission, firms must satisfy the "threshold conditions" both on authorization and on an on-going basis. The conditions require that the applicant: 1) Has its head office in the UK or it carries on business in the UK? 2) Is adequately capitalized to conduct the reinsurance business in question? 3) Has appropriate management systems and controls in place? 4) Has suitably qualified persons that are "fit and proper" and capable of performing senior management functions?

There are some special components in the UK reinsurance regulation. One of them is the notification obligation that the cedant company has to the reinsurance company. Under the Insurance Act 2015, a reinsurance contract will be classified as a "non-consumer insurance contract" meaning that: 1) The cedant's disclosure obligation will be re-characterized as a "duty of fair presentation". 2) The cedant's fair presentation of the risk must be reasonably clear and accessible to a prudent reinsurer. 3) The cedant must disclose every material circumstance that it knows or ought to know or, failing that, must give the reinsurer sufficient information to put a prudent reinsurer on notice that it needs to make further enquiries.

The Insurance Act 2015 brings a lot of notable changes to a cedant. This Act introduces a new concept that the cedant ought to know all available "information" through rational search before being a cedant.

3.3.2 Switzerland
Switzerland provides a good environment for insurance and reinsurance business. These institutions are subject to the supervision of the Swiss Financial Market Supervisory Authority (FINMA).

The Federal Insurance Supervision Act (ISA) provides regulatory requirements for insurance and reinsurance undertakings in Switzerland. An insurance group consists of two or more companies should meet the following rules: 1) At least one insurance company is in the group. 2) The companies mainly engaged in the field of insurance. 3) The companies form an economic unit or are otherwise tied to each other through influence or control.

An insurance conglomerate is similar to an insurance group but has one extra rule: it should have at least one bank or securities dealer of major economic importance. The concept of insurance in Swiss law rests on the following five elements: risk or danger, performance by the insured, performance by the insurer, independence of the operation, and compensation of risks according to the laws of statistics. Although the basic regulatory requirements of insurers and reinsurers are the same, there are still some points worth noting: 1) The minimum capital requirements differ depending on the scope of insurance activities. 2) There are some supplementary provisions only applying to certain types of insurance activities. 3) Additional requirements are applied to foreign insurance undertakings.

3.3.3 Denmark

The authorized supervisory body in Denmark is the Danish Financial Supervisory Authority (DFSA). The DFSA is responsible for supervising the financial sector including insurance companies. Insurance or reinsurance companies are required to have a license for doing business in Denmark. The DFSA considers its most important supervisory task to be ensuring that financial undertakings have sufficient funds to cover their own risks and solvency.

Insurance companies from other European Economic Area (EEA) countries may do insurance business in Denmark but they are required to comply with the Freedom-of-Services Regulation. However, Insurance and reinsurance companies from non-EEA countries are in general not able to carry out insurance business in Denmark.

3.3.4 Netherlands

In the Netherlands, De Nederlandsche Bank (DNB) is the supervisor of insurance and reinsurance companies. The Financial Supervision Act (FSA) is the statutory instrument to regularize the behaviors of insurers and reinsurers. In addition, there are some secondary legislations related to the FSA such as the Decree on Prudential Rules (DPR) of the FSA, the
Decree on Supervision of Conduct (DSC) of Financial Entities of the FSA, and the Decree on Market Access (DMA) of the FSA.

Any companies carrying on business of insurance and reinsurance must apply license granted by DNB. In order to have the license, insurance companies should submit a business plan describing the nature and internal risks of the insurance business. The plan must contain proof that the insurer either meet the minimum guarantee fund requirement or the solvency margin requirement. In addition, the following information is required as well: a description of the administrative organization, proof of the financial means, and the financial estimates of the first three book years. FSA and DPR mainly focus on regulations with respect to the following issues: trustworthiness, integrity and sound management, controlled pursuit of business, outsourcing activities, minimum funds and solvency.

3.4 Pacific-Australia

In Australia, general insurers and life insurers, together with banks, credit unions, building societies, friendly societies, and most of the superannuation industry are primarily regulated by the Australian Prudential Regulation Authority (APRA). APRA is responsible for managing the governing Acts and setting and implementing efficient standards to most regulated entities.

Required by APRA, reinsurers need to have a reinsurance management framework so that they have enough confidence to show that reinsurance arrangements are well managed. The framework must demonstrate reinsurance management strategy that includes the key elements, policies, procedures and controls of the insurer. Besides that, a good measurement of a well-established framework is that it also provides sufficient consideration of business classes, complexity of the insurer's operations and its risk appetite.

In addition, APRA also requires the insurer to submit a document called "Reinsurance Statement" on an annual (or six-monthly) basis. "Reinsurance Statement" provides detailed information of the insurer's reinsurance arrangement. The insurer must achieve legally binding reinsurance arrangements and reach a minimum as required by "two-month rule" and the "six-month rule" in GPS230 by APRA.

3.5 Asia

3.5.1 China

In China, the reinsurance industry is a relatively new business because there was no professional domestic reinsurance before 1986. And the reinsurance department was founded within the People's Insurance Company of China (PICC). The first domestic reinsurer in China was
founded in 1999. Four years later, Munich Re Beijing Branch became the first licensed foreign reinsurer, and all the major reinsurers worldwide started to have business in China gradually.

The contract of reinsurance is a special type of insurance contract concluded between the insurer and the reinsure, and regulated by the Insurance Law of the People's Republic of China. Due to the law regulation, there is a minimum capital requirement required for any insurance company including reinsurance. Solvency requires the company to have a minimum capital, and the insurance law also requires the premium/surplus ratio and One Tenth risk tension ration rule.

The China Insurance Regulatory Commission require the direct insurance company to submit the materials: where any direct property insurance business is ceded in the manner of proportional reinsurance, except for aerospace insurance, nuclear insurance, petroleum insurance, and credit insurance, if the company conducts treaty reinsurance and facultative reinsurance in the prior fiscal year, it shall submit information on its transactions in which the business ceded to the same reinsurer for each risk unit exceeds 50% of the insured amount or the limit of liability in the direct insurance contract underwritten by the cedant.

3.5.2 Malaysia

The reinsurance industry develops rapidly in Malaysia. In order to expedite the economic development of the country, especially the island of Labuan, the Malaysian government has opened its doors to global brokers and reinsurers.

For the overseas reinsurers who applied for and were granted licenses under the Malaysian Insurance Act to operate in Malaysia, they are also expected to develop and bring in off-shore or non-Malaysian business to assure that their operations in the country are regional in scope.

As for the regulation, Bank Negara Malaysia (BNM) regulates entities which carry on insurance and reinsurance business, insurance broking, adjusting and financial advisory. Insurers are licensed by the Minister of finance on the recommendation of the BNM. Brokers, adjusters and financial advisers are licensed directly by BNM.

There are some guidelines issued by BNM, the reinsurance supervisory authority in Malaysia: 1) The reinsurer must be legally set up in accordance with the laws of its home country and has been authorized to carry on reinsurance business in other countries and Malaysia is not precluded. 2) The use of various tools and publications to assess the capacity and financial strength of the reinsurer. In the case of overseas placements, insurers must ensure that the reinsurers they use for such placements must have a minimum of "A" rating by an accredited rating agency or have a combined paid-up capital and surplus of at least USD 150 million. 3) Total reinsurance cessions (facultative and treaty) to foreign reinsurers should not exceed 50% of the direct-writing company's total reinsurance premium. 4) No one foreign reinsurer shall hold more than 25% of a
risk in the case of a lead reinsurer and 10% of a risk in the case of other participants. 5) In
general, insurers shall ensure that their reinsurance arrangements fall in line with national
aspirations and to the extent possible, accord priority to optimization of the Malaysian insurance
capacity followed by Labuan, before securing foreign reinsurance support.

In Malaysia, all insurers and reinsurers, licensed under the Insurance Act 1996 should act within
the Risk-Based Capital framework for business generated inside and outside of the country. The
Supervisory Target Capital Level set by Bank Negara Malaysia (BNM) is 130%. The Individual
Target Capital Level set by each insurer reflects the risk profile and must be higher than 130%.

3.5.3 Japan

In Japan, reinsurance is essential for the insurance market because of the expansion
diversification and complication of risk. Especially, Japan experiences many earthquakes and
typhoons due to its geographic position. Therefore, all insurance companies are required to
reinsure and transfer these risks to a foreign reinsurer.

The Prime Minister of Japan is the dominant regulator of the insurance market under the
Insurance Business Act (IBA). Other than significant powers like granting and cancelling
insurance business licenses, most have been assigned to the Commissioner of the Financial
Services Agency (FSA) of the Japanese government and to the directors of the Local Finance
Bureau and the Local Finance Branch Bureau of the Ministry of Finance (collectively LFB).

There are minimum capital requirements for different companies: licensed insurance companies:
¥1 billion capital; small amount and short-term insurance business (SASTI) insurers: ¥10 million
capital plus ¥10 million x (or more) deposit; licensed foreign insurers: ¥200 million deposit;
insurance brokers ¥20 million (or more) deposited guarantee.

3.5.4 India

The Insurance Act, 1938, the Insurance Regulatory and Development Authority Act, 1999, the
Insurance Regulatory and Development Authority of India (Authority), and the Insurance
Advisory Committee altogether regularize the insurance and reinsurance market in India.

Regulations in India have strict requirements for objectives and procedures of reinsurance
arrangements. Other than that, the insurers/reinsurers in Indian must ensure that their reinsurance
arrangements are able to deal with catastrophe accumulations through realistic disaster scenario
testing methods. They also need to file with the Authority any new reinsurance arrangement and
a copy of every reinsurance treaty contract wordings and excess of loss cover note within the 30
days of Commencement of the financial year. Cross border and domestic reinsurance are under
different criteria as well. For inward reinsurance business, they need to have a well-defined
underwriting policy approved by the Board of Directors, file with the authority its underwriting policy, ensure that the decisions are made by professional persons, and file with the authority any official changes to the underwriting policy. In addition, outstanding loss provisions are also needed for every insurers and reinsurers in India.
Chapter 4

Reinsurance Products and Methods

Just like insurance, reinsurance can be written for almost any type of risk. This chapter will introduce nine common reinsurance products and common industry practices to write reinsurance contracts for those risks. Table 4 below lists the reinsurance products sold by several leading reinsurance companies according to AM Best and is meant to acquaint the reader to the various reinsurance products existing in the market.

<table>
<thead>
<tr>
<th>Table 4 Reinsurance Products</th>
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<tbody>
<tr>
<td>Munich Reinsurance Company</td>
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<tr>
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<td>Swiss Reinsurance Company Limited</td>
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<td>Hannover Rueckversicherung AG</td>
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<tr>
<td>SCOR S.E.</td>
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<tr>
<td>Company</td>
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<td>---------------------------------</td>
</tr>
</tbody>
</table>
| Allianz S.E.                    | • Credit and Surety  
• Inherent Defects Insurance (IDI)  
• Space  
• Agriculture  
• Marine & Energy  
• Aviation  
• Engineering  
• US Property CAT |
| Everest Re Group Ltd.           | P & C (treaty and facultative):  
• Life & Health  
• Employee Benefits  
• CAT  
• Credit & Bond  
• Agricultural |
| Korean Reinsurance Company       | Treaty Property:  
• Catastrophe Treaties  
• Retrocession Catastrophe Treaties  
Treaty Casualty  
• Workers’ Compensation  
• General Liability  
• Auto Liability  
Facultative Property  
• Primary and Excess of Loss  
Facultative Casualty  
• Directors and officers  
• Miscellaneous Professional Liability  
• Accident & Health  
• Marine & Aviation  
• Surety |
| General Insurance Corporation of India | Non-life Insurance  
• Property  
• Engineering  
• Marine  
• Casualty  
Life Insurance  
• Insurance Against Death  
• Health Insurance |
|                                | Property (Facultative, Treaty)  
• Marine Hull, Cargo & Offshore Energy  
• Aviation (Airlines, General Airlines, Treaty (Prop))  
• Liability (Facultative, Treaty Proportional, Non-Proportional) |
<table>
<thead>
<tr>
<th>Company</th>
<th>Product Lines</th>
</tr>
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<tbody>
<tr>
<td>XL Group plc</td>
<td><strong>Life (Proportional; Non-proportional)</strong></td>
</tr>
<tr>
<td></td>
<td><strong>Casualty</strong></td>
</tr>
<tr>
<td></td>
<td>• Casualty Facultative (Primary, excess, umbrella)</td>
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<tr>
<td></td>
<td>• Casualty Treaty (Proportional, non-proportional, excess of loss, quota share, surplus)</td>
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<td><strong>Property</strong></td>
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<td></td>
<td>• Property Catastrophe</td>
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<td>• Property Treaty (excess of loss, proportional, quota share)</td>
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<tr>
<td></td>
<td>• Property Facultative (Non-proportional, primary, Proportional, Catastrophe and Non-Catastrophe, High Excess and Working exposures)</td>
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<tr>
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<td>• Property Risk Retrocession</td>
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<tr>
<td>The Toa Reinsurance Company Limited</td>
<td><strong>Treaty Property (Proportional/Excess of Loss)</strong></td>
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<tr>
<td></td>
<td>• Catastrophe</td>
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<td></td>
<td>• Aggregate excess of loss; specialty</td>
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<td></td>
<td><strong>Casualty (Excess of Loss)</strong></td>
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<td></td>
<td>• Treaty Casualty Clash Cover Professional Liability</td>
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<td></td>
<td>• Casualty Facultative</td>
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<tr>
<td>Caisse Centrale de Reassurance</td>
<td><strong>Non-life/P&amp;C Reinsurance</strong></td>
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<tr>
<td></td>
<td>Traditional life, accident and health reinsurance</td>
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<td></td>
<td>Specialty lines</td>
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<tr>
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<td>• Credit &amp; Surety</td>
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<td>• Political Risk</td>
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<td>• Non-life CAT</td>
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<td>Renaissance Re Holding Ltd.</td>
<td><strong>Property</strong></td>
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<td>• Cat Excess of Loss Reinsurance</td>
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<td>• Per Risk Excess of Loss</td>
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<td>• Proportional (quota share and surplus share)</td>
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<td>Casualty &amp; Specialty (Excess of Loss/Quote Share)</td>
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<td>Company</td>
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| ACE Limited                    | P&C (Excess of Loss/Quote Share); | • General Liability  
• Umbrella  
• Environmental Liability  
• Personal Automobile  
• Property/Multi-line  
• Surety and Fidelity  
• Workers Compensation  
• Commercial Auto  
• Alternative Risk and Casualty |
| Amlin plc                      | Property & Casualty             | • Risk Excess of Loss  
• Catastrophe Excess of loss  
• Proportional  
• Specialty (Engineering; Surety & Credit, Motor, Agriculture & Crop, Structured Solutions, Marine)  
• Aviation & Satellite  
• Nuclear  
• Contingency  
• Terrorism |
| Alterra Capital Holdings Ltd.  | Property & Casualty             | • Property Treaty CAT (per risk, quota share)  
• Casualty (general casualty, medical malpractice, professional liability, environmental liability, automobile liability)  
• Specialty (surety, credit and political risk, mortgage, marine, energy, commercial airline hull and liability, general aviation)  
• Casualty Facultative (automobile liability)  
• General Liability (products and umbrella)  
• Miscellaneous Professional Liability  
• Public Entity |
| IRB-Brasil Resseguros S.A      | Property & Casualty             | • General Liability  
• Environmental  
• Professional  
• Aviation  
• Miscellaneous  
• Engineering  
• Surety  
• Financial Lines  
• Oil & Gas  
• Agricultural risk |
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<th>Treaty/Reinsurance Products</th>
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<td>Facultative &amp; Specialty Reinsurance Treaty reinsurance</td>
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<td>Allied World Assurance Company Holdings</td>
<td>Property &amp; Casualty</td>
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<td>• General Casualty Treaty</td>
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<td>• Marine &amp; Aviation Treaty</td>
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<td>• Professional Lines Treaty</td>
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<td>• Specialty Lines Treaty</td>
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<td>W.R. Berkley Corporation</td>
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<td>• Professional Liability Non-Healthcare</td>
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<td>• Property</td>
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### 4.1. Property

Property insurance provides protection against risks to property such as fire, theft, earthquake, etc. Due to its nature of protecting structures, writing property insurance results in a high maximum possible loss. Property reinsurance normally covers the higher-end of the claims spectrum that are rare but devastating for an insurer. Also, these contracts are normally written on a per-risk basis due to the difference in risk profiles for each building. A common product that falls under property is the excess-of-loss catastrophe coverage that protects the ceding company from losses due to natural disasters such as floods or hurricanes. Another reinsurance product falling under property is engineering, which covers risks happening during the construction phase.

### 4.2. Workers' Compensation

Workers' compensation insurance provides wage replacements and medical benefits to employees injured during employment. In some countries such as the United States, employers are required to provide workers' compensation to their employees. According to the Insurance Information Institute, insurers focus more on loss frequency rather than severity while charging premiums because major accidents are bound to occur more frequently in a dangerous work environment. These various work environments are assigned class codes that assist reinsurers in pricing. According to Willis Re, workers' compensation contracts are normally written on an
excess-of-loss basis. However, an aggregate excess-of-loss protection can be added to the original contract to protect ceding companies from catastrophe, such as terrorism.

4.3. Professional Liability

Professional liability insurance is written to protect professionals against liability incurred because of errors when performing their professional services. Liability might involve financial loss that the insured's client experiences due to the error and legal costs charged on the professional because of getting sued. This type of insurance is considered complicated to write due to having a heterogeneous claims group. For example, the amount of risk associated with an accountant may be greatly different from the risk associated with a construction consultant. Hence, reinsurance practices vary depending on the underlying risk associated and products can be written on a quota share or excess of loss basis. Products will normally have an upper limit associated and they are normally written on a per-risk basis but some reinsurers also offer treaty contracts. Some common sub-classes of this product are errors and omissions (E&O), directors and officers (D&O), and medical malpractice.

4.4. Motor

Motor insurance provides coverage for third-party liability in an accident and/or physical damage to the car. According to SCOR and the World Bank, motor insurance is deemed to be non-life insurance's greatest source of premiums. In pricing motor insurance, drivers are separated into risk classes that are usually homogenous. An interesting case with motor insurance though is when insuring the third-party liability because the minimum insurable value varies in different countries and this factor may determine if insurers would like a non-proportional coverage, proportional coverage, or a combination of both. Developed countries are placing more emphasis on covering bodily injury than emerging markets and hence, non-proportional coverage or a combination of proportional and non-proportional coverage. Insurers in emerging markets may not even purchase non-proportional coverage due to a lower minimum coverage for third party motor liability. Motor reinsurance can be written on a per occurrence, per risk basis or on an aggregate risk basis which looks at the entire portfolio and motor reinsurance contracts are normally written as a treaty contract while some facultative contracts still do exist.

4.5. Marine

Marine insurance protects shipping and cargo companies from major losses in case of transportation accidents that result in the loss of cargo and damages to the ship carrying cargo. Marine insurance is one of the earliest developed insurance products in the world with a history starting from shipping contracts written in Medieval Italy followed by the meeting of shipping
insurers at Lloyd's Coffee House in the late 1680s London. Marine insurance is normally divided between cargo, which protects the shipment carried, and hull, which protects the ship itself. Other products included under the class of marine insurance are offshore energy, marine liability, yacht, ports and terminals, and specie and fine arts. Marine reinsurance is normally written on an excess-of-loss basis.

4.6. Agriculture

Agriculture or crop insurance protects farmers from production risks and financial risks associated with crops. Common products associated with crop insurance includes the multiple peril crop insurance, which is a federally supported product that covers crop losses caused by natural events such as a drought or insect damage, crop-hail insurance, which protects farmers from losses due to hail, and crop revenue insurance, which protects farmers from low yields and low prices. Usually, crop reinsurance is written on a stop-loss basis.

4.7. Life

Life insurance protects a policyholder's beneficiary from financial loss that results from the unexpected loss of the policyholder's life. According to Transamerica Reinsurance, writing a life reinsurance contract is in many ways different from writing a P&C reinsurance contract. In terms of risk characteristics, life has a small risk concentration as opposed to P&C. To illustrate, if a hurricane strikes, people can pre-emptively evacuate the hurricane zone but buildings cannot move resulting in a much higher claim for property as opposed to life. Also, life is a long-tailed product while P&C can be either short or long. Most likely, the claim amount for life is known while for P&C, it remains unknown until the incident. Then, premium rates for life insurance are fixed while premiums change per year for most P&C products. Because of these differences, the life side focuses more towards the asset side (optimizing capital) while P&C focuses on the liability side. Hence, reinsurers need to take a different approach in writing life reinsurance as opposed to P&C.

Life insurances typically buy reinsurance to share mortality/morbidity risk, lapse risk, investment risk, and limit catastrophe risk. The four common ways to write life reinsurance is through the yearly renewable term (YRT), coinsurance, modified coinsurance, and coinsurance with funds withheld. In YRT contracts, only mortality or morbidity risk is transferred and reinsurers will normally have predetermined rates based on the risk class of the insurers' policyholders that can be further negotiated. After rates are set, they are rarely changed. In a coinsurance contract, the reinsurer shares the insurer's portfolio in terms of both premiums and claims percentage, making this contract really similar to a quota-share contract. The insurer will pay the reinsurer a premium equal to the premium that the insurer charges the policyholder. In return, the reinsurer will cover administrative expenses and agent commission of the ceding company. The reinsurer will also
need to set up reserves for its portion of the portfolio. This method is used so that insurers can sell more products and manage their capital better. A modified coinsurance contract works in the same way as a coinsurance contract except that the ceding company is still responsible to hold reserves for the portfolio transferred to the reinsurer. This method gives the ceding company more control over its assets and reduces its credit risk issues. In coinsurance with "funds withheld", the ceding company keeps the assets but the reinsurer needs to set up reserves. However, the assets initially kept by the ceding company "belongs" to the reinsurer which is why the reinsurer has an asset represented as "funds withheld". That same asset can be seen in the liability section of the ceding company.

According to RGA Reinsurance Company, life reinsurance contracts used to be mainly written on a proportional basis. However, Solvency II is changing how life reinsurance contracts will be written. RGA assesses that stop-loss treaties will be more effective in a ceding company’s capital management strategy when the Solvency II ratio is applied. One product proposed is a "shock absorber" product, which is an excess-of-loss reinsurance contract in which the reinsurer pays the ceding company if the lapse or longevity rate increases beyond a certain threshold.

4.8. Accident and Health

Accident and Health insurance protects policyholders from unforeseen losses caused by unexpected events and expensive hospital bills. Health expenditure has been rapidly rising over the past years to the point that healthcare becomes too expensive. Under the Affordable Care Act, insurers were not allowed to deny insurance, nor charge higher rates to people with pre-existing conditions that may lead to an increase in premiums. To ensure insurance premiums remain low enough, several states such as New York, Arizona, and Connecticut, created their own reinsurance programs. Some states write health reinsurance on a surplus share basis and some on a stop-loss basis.

4.9. Specialty Products

Reinsurance companies also reinsure special risks with low frequency and high severity such as space and aviation and they cover risks that primary insurers might otherwise exclude such as TNCB (Terrorism, Nuclear, Chemical, and Biological) risks. Various reinsurers, organized by a broker, will pool resources together to insure these types of risks due to its low frequency and high severity. Hence, these reinsurance contracts are multi-layered non-proportional or proportional contracts.
Chapter 5

Reinsurance Optimization Problems

In early optimal reinsurance strategies, variance was the most common measure of risk used. Borch (1960) used the variance risk measure to prove that a stop-loss reinsurance treaty is the optimal reinsurance strategy. In addition, Borch (1969) noted that while a stop-loss treaty is the best reinsurance strategy to minimize a ceding company's variance for the least net premium paid, a stop-loss treaty also results in a large variance within the ceded claims transferred to the reinsurer. Hence, assuming the reinsurer charges the same loading parameter for a quota-share and excess-of-loss contract, a rational reinsurer will always choose to sell quota-share contracts instead of stop-loss contracts. Furthermore, claims distributions in reality are rarely symmetrical and hence, variance may not be the best risk measure to use in negotiating reinsurance contracts. From that conclusion, Borch noted that research into optimal reinsurance strategies should consider other goals other than minimizing variance, such as the utility of the ceding company.

In recent years, there are several models reinsurance companies used to determine the optimal reinsurance ceded function, such as minimizing the ruin probability, the variance, or risk measure and maximizing the expected discount factor, the joint survival probability, or expected utility.

On top of that, various premium principles such as Wang's premium principles that consider layers of reinsurance, was developed. These new optimal reinsurance strategies are developed with more consideration to real-life practices and the variety of problems that optimal reinsurance seeks to solve make research into optimal reinsurance strategies interesting.

Suppose $X$ is the total risk/claim amount faced by the insurer and $I(x)$ is the ceded function we will adopt to cede part of the risk to the reinsurer. Thus, the total ceded loss is $I(X)$, and the retained loss by the insurer is $X-I(X)$. Then, the optimal reinsurance strategies are defined by $I^*$, which are usually optimizers of certain optimization problems. Note that parameters utilized in this paper are listed on next page.
Symbol list

The following notation will be used across this paper:

\( \alpha \) \hspace{1cm} Quota share/Proportion
\( m \) \hspace{1cm} Excess of loss point
\( d \) \hspace{1cm} Deductible
\( X \) \hspace{1cm} Total claim amount
\( I \) \hspace{1cm} Ceded loss function
\( R_i \) \hspace{1cm} Retained loss/Retension
\( I^* \) \hspace{1cm} Optimal reinsurance policy
\( C \) \hspace{1cm} Insurer's premium
\( \Pi \) \hspace{1cm} Reinsurer's premium
\( \pi \) \hspace{1cm} Reinsurance premium budget
\( T_i \) \hspace{1cm} Total risk exposure for Insurer
\( \eta \) \hspace{1cm} Insurer's safety loading parameter
\( \theta \) \hspace{1cm} Reinsurer's safety loading parameter
\( u \) \hspace{1cm} Initial reserve for insurer
\( \lambda \) \hspace{1cm} Rate of the Poisson distributed random variable
\( t \) \hspace{1cm} Time frame
\( \phi \) \hspace{1cm} Ruin probability function
\( U \) \hspace{1cm} Utility function
\( \psi \) \hspace{1cm} Wealth function
\( A \) \hspace{1cm} Rate of risk aversion
\( Var \) \hspace{1cm} Variance
\( r \) \hspace{1cm} Rate of interest
5.1 Minimizing Risk

Most of the time, optimal reinsurance problems involve the best way to minimize risk under certain constraints of premium principles and ceded functions. Minimizing risk is important for an insurance company because it reduces adverse risk exposure and allows the insurance company to write more business. There are a couple of risk measures commonly used in optimal reinsurance papers: variance, standard deviation, VaR, TVaR and ruin probability. This section aims to summarize and discuss past papers written on optimal reinsurance that aims in minimizing risk.

The following premium principles are commonly used for either the insurer or the reinsurer:

(a) Expectation principle: $\Pi(X) = (1 + \theta)E[X], \theta > 0$;
(b) Standard deviation principle: $\Pi(X) = E[X] + \beta \sqrt{Var[X]}, \beta > 0$;
(c) Variance principle: $\Pi(X) = E[X] + \beta Var[X], \beta > 0$;
(d) Modified variation principle: $\Pi(X) = E[X] + \beta \sqrt{Var[X]} + \gamma Var[X]/E[X], \beta, \gamma > 0$;
(e) Wang’s premium principle: $\Pi(X) = (1 + \theta) \int_0^\infty g(S_X(x))dx$, where $\theta \geq 0$ is the safety loading of the reinsurer, and $g: [0, 1] \rightarrow [0, 1]$ is an increasing and concave function with $g(x) \geq x, g(0) = 0$, and $g(1) = 1$;
(f) Dutch premium principle: $\Pi(X) = E[X] + \theta' E[(X - \gamma E[X])_+], \gamma \geq 1, 0 < \theta' \leq 1$;
(g) Mixed principle: $\Pi(X) = E[X] + \beta Var[X]/E[X]$, where $\beta > 0$;
(h) p-mean value principle: $\Pi(X) = (E[X^p])^{1/p}$, where $p > 1$;
(i) Semi-deviation principle: $\Pi(X) = E[X] + \beta E(X - E[X]_+)^2, \theta > 0$ and $X'$ is an independent copy of $X$;
(j) Gini principle: $\Pi(X) = E[X] + \beta E[X - X'], \beta > 0$ and $X'$ is an independent copy of $X$;
(k) Generalized percentile principle: $\Pi(X) = E[X] + \beta [F^{-1}_X (1 - p) - E[X]], \beta > 0$ and $p < 1$;
(l) CTE principle: $\Pi(X) = \frac{1}{p} \int_0^1 F^{-1}_X(x)dx$, where $0 < p < 1$;
(m) Semi-variance principle: $\Pi(X) = E[X] + \beta E(X - E[X])^2, \beta > 0$;
(n) Quadratic utility principle: $\Pi(X) = E[X] + \gamma - \sqrt{\gamma^2 - Var[X]}, \gamma > 0$ and $\gamma^2 \geq Var[X]$;
(o) Covariance principle: $\Pi(X) = E[X] + 2\beta Var[X] - \beta Cov(X, Y)$, where $\beta > 0$ and $Y$ is a random variable;
(p) Exponential principle: $\Pi(X) = \frac{1}{\beta} logE[exp(\beta X)], \beta > 0$.

5.1.1 Under Variance or Standard Deviation

...
5.1.1.1 Maximize the Total Reduction in Claim Portfolio

Benktander (1975) looked into the optimal reinsurance problem from both the insurer and reinsurer's perspectives to maximize the total reduction in a claims portfolio, which is defined by the variance of the original portfolio subtracted by the variance of the ceded function and the variance of the retained function. While there is no formal objective function defined in the paper, it can be mathematically defined as

\[ \text{Var}[X] - \text{Var}[I(X)] - \text{Var}[R_t(X)] > 0. \]

A quota share reinsurance strategy is found to be the most optimal reinsurance strategy to achieve the goal. An excess-of-loss treaty though will benefit the ceding company more than the reinsurer as the reinsurer will be exposed to a higher variance than the ceding company. This degree of riskiness increases as a distribution function becomes more skewed. For example, a reinsurer will be exposed to more variance than the ceding company when reinsuring claims with a Pareto distribution of 2 degrees of freedom as opposed 5 degrees of freedom. Also, the reduction in total variance by excess-of-loss decreases as a distribution function becomes more skewed. Hence, the author questions the adequacy of an excess-of-loss contract to reinsure dangerous portfolios.

5.1.1.2 Minimize the Variance of the Reinsurer's Risk

For minimizing risk under the variance risk measure, Gajek and Zagrodny (2000) developed an optimal reinsurance strategy to minimize the variance of an insurer's retained risk, \( R_t(X) \). The objective function can be expressed as

\[ \min \text{Var}[R_t(X)] \]

over the space of all measurable functions \( I: \mathbb{R}_+ \to \mathbb{R} \), where \( \mathbb{R}_+ = [0, \infty) \), subject to the constraint

\[ 0 \leq R_t(X) \leq X. \]

The premium principle, \( \Pi(X) \), used is the standard deviation premium principle and it must be less than the ceding company’s budget for reinsurance \( \pi \). The premium charged is defined as

\[ \Pi(X) = E[I(X)] + \theta \sqrt{\text{Var}[I(X)]} \leq \pi, \]

where \( \theta \) is the loading factor that the reinsurer uses, and \( I(X) \) is the amount of claims ceded to the reinsurer. The authors also assume \( E[X^2] < \infty \).
The change-loss reinsurance strategy is determined to be the optimal reinsurance strategy for this problem, in which the reinsurer reimburses a constant percentage of the claim, $1 - a$, above m, the change-loss point (the agreed amount where reinsurance starts to apply). The optimal ceded function is determined by

$$I(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq m, \\ (1 - a)(x - m)_+, & \text{o.w.} \end{cases}$$

where $a$ and $m$ are solutions to

$$\begin{cases} E[X] - m - \int_m^\infty (x - m) dF + \frac{r}{\theta} \sqrt{\int_m^\infty (x - m)^2 dF} - \left( \int_m^\infty (x - m) dF \right)^2 = 0, \\ (1 - a) \int_m^\infty (x - m) dF + \theta \sqrt{\int_m^\infty (x - m)^2 dF} - \left( \int_m^\infty (x - m) dF \right)^2 = \pi. \end{cases}$$

5.1.1.3 Mean and Variance of the Reinsurer's Share of Total Claim Amount

Kaluszka (2001) derived optimal reinsurance under premium principles based on the mean and variance of the reinsurer's share of the total claim amount. The premium principles applied in this paper:

- Expected value principle: $\Pi = (1 + \beta) E[I(X)], \beta > 0$
- Standard deviation principle: $\Pi = E[I(X)] + \beta \sqrt{Var[I(X)]}, \beta > 0$
- Variance principle: $\Pi = E[I(X)] + \beta Var[I(X)], \beta > 0$
- Modified variance principle: $\Pi = E[I(X)] + \beta \frac{Var[I(X)]}{E[I(X)]}, \beta > 0$

The above principles are of the form:

$$E[I(X)] = f(\Pi, \sqrt{Var[I(X)]}),$$

where

$$f: (0, \infty) \times [0, t(\Pi)) \to (-\infty, \infty) \text{ with } \sqrt{Var[X]} < t(\Pi) \leq +\infty,$$

and the following conditions hold:

- $f(\Pi, 0) = \Pi$;
- $f(\Pi, t)$ is nonincreasing, concave, and differentiable in $t$.

Note that $I(X)$ is optimal if it satisfies:

$$\min \sqrt{Var[X - I(X)]}$$

subject to
\[
E[I(X)] = f \left( \Pi, \sqrt{\text{Var}[I(X)]} \right)
\]

and

\[
0 \leq I(x) \leq x \text{ for all } x \geq 0 .
\]

Both global reinsurance and local reinsurance are studied in this paper as illustrated below.

- **Global Reinsurance**

Let \( N \) be the number of claims over that period. The goal is

\[
\min_{I \in \mathcal{I}} \text{Var}[X - I(X)]
\]

where \( I(f) = \{ I \mid E[I(X)] = f \left( \Pi, \sqrt{I(X)} \right), 0 \leq I(x) \leq x \text{ for every } x \geq 0 \} \).

If \( E(X) > f \left( \Pi, \sqrt{I(X)} \right) \), the **general optimal ceded functions** are of the following form:

\[
I^*(x) = a(x - b)_+ ,
\]

where \( a \) and \( b \) are solutions to

\[
-E(b - X)_+ f'_2 \left( \Pi, a \sqrt{\text{Var}[(X - b)_+]} \right) = (1 - a) \sqrt{\text{Var}[(X - b)_+]}
\]

and

\[
f \left( \Pi, a \sqrt{\text{Var}[(X - b)_+] \right) = aE[(X - b)_+]
\]

with

\[
0 < a \leq 1, 0 \leq b \leq \sup (X)
\]

and

\[
f'_2(\Pi, t) = \partial f(\Pi, t)/\partial t .
\]

He then demonstrated that above formula by applying the expected value principle, standard deviation principle, the variance principle, and the modified variance principle.

- **Local Reinsurance with different constraints**

The first optimization problem is

\[
\min_{I \in \mathcal{I}} \text{Var} \left[ \sum_{i=1}^{N} (X_i - I(X_i)) \right]
\]

with \( I(f) = \{ I \mid E[I(X)] = f \left( \Pi, \sqrt{\text{Var}[I(X)]} \right), 0 \leq I(x) \leq x \text{ for every } x \geq 0 \} \).

If

\[
E[X] > \Pi \text{ and } -f'_2(\Pi, t(0)) \sqrt{\text{Var}[I(X)]} \leq \frac{E[N] \sqrt{\text{Var}[X]}}{\text{Var}[N] E[X]},
\]

where \( t(0) \) is a real such that

\[
t(0)E[X] = f \left( \Pi, t(0) \sqrt{\text{Var}[X]} \right),
\]

the **general optimal ceded functions** are of the following form:

\[
I^*(x) = a(x - b)_+ .
\]

where \( 0 < a \leq 1, 0 \leq b \leq \sup (X) \) are solutions to
\[\gamma'_2(b, a) = 0\]
\[aE[(X - b)_+] = f(\Pi, a\sqrt{\text{Var}[(X - b)_+]})\]

with

\[
\gamma(s, t) = E[N]\text{Var}[X] + (t^2 - 2t)\text{Var}[(X - s)_+] - 2E[(s - X)_+]f(\Pi, t\sqrt{\text{Var}[(X - s)_+]}) \\
+ \{E[X] - f(\Pi, t\sqrt{\text{Var}[(X - s)_+]})\}^2\text{Var}[N], \text{ for } 0 \leq t \leq 1, s \geq 0
\]

and

\[\gamma'_2(s, t) = \frac{\partial \gamma(s, t)}{\partial t}.\]

He then demonstrated the optimal ceded function using standard deviation principle, and relaxed his constraint above that \(0 \leq E[I(X)] \leq E[X]\).

The second optimization problem is

\[
\min_{I \in I_E(f)} \text{Var}[\sum_{i=1}^{N}(X_i - I(X_i))]
\]

with

\[I_E(f) = \{I|0 \leq E[I(X)] \leq E[X], E[I(X)] = F(\Pi, \sqrt{\text{Var}[I(X)]})\}.
\]

The optimal ceded function for local reinsurance is

\[I^{**}(x) = a(x - E[X]) + f(\Pi, a\sqrt{\text{Var}[X]}),\]

where \(a\) is the minimizer to

\[(1 - t)^2E[N]\text{Var}[X] + (E[X] - f(\Pi, t\sqrt{\text{Var}[X]}))^2\text{Var}[N]\]

under constraint \(0 \leq f(\Pi, t\sqrt{\text{Var}[X]}) \leq E[X]\). He then demonstrated the optimal ceded function using standard deviation principle.

**5.1.1.4 Under Absolute Deviation and Truncated Variance**

Gajek and Zagrodny (2004) discussed the best risk protection from a reinsurance company and derived the forms of optimal contracts under the absolute deviation and truncated variance risk. They defined that \(I\) is the contract between the insurer and the reinsurer and \(\rho(I)\) is the expected harm. The class \(\mathcal{R}(I_1, I_2)\) of all measurable functions

\[I: [0, \infty) \to (-\infty, \infty)\]
for which the inequalities $I_1(X) \leq I(X) \leq I_2(X)$ hold for all $X \geq 0$. In addition, $\mathcal{R}_0$ is defined by the boundary functions

$$I_1(x) \equiv 0 \text{ and } I_2(x) \equiv x,$$

and $\varphi: \mathbb{R} \rightarrow \mathbb{R}_+$ is a harm function that measures the insurer’s loss,

$$\rho(I) = E[\varphi(X - I(X) - E[X - I(X)])].$$

They assumed that the insurer is interested to purchase a contract $I^*$ such that $\Pi(I^*) \leq \pi$ and $\rho(I^*) \leq \rho(I)$ for all $I \in \mathcal{R}$ which has premium $\Pi(I(X)) \leq \pi$, where $\mathcal{R}$ is the set of all admissible contracts. A contract $I$ is considered optimal if it satisfies the above conditions.

They first considered the general sufficient conditions for optimality: $\varphi$ is a measurable function.

The premium is defined by **standard deviation principle**:

$$\Pi = E[I(X)] + \beta \sqrt{\text{Var}[I(X)]}, \beta > 0,$$

and $\varphi: \mathbb{R} \rightarrow \mathbb{R}_+$ a measurable harm function, he assumed the following conditions hold throughout the paper:

(a) $E[X] < \infty$,
(b) $E[I_1(X)^2] < \infty$ and $E[I_2(X)^2] < \infty$,
(c) $E[\varphi(X - I(X) - E[X - I(X)])] < \infty$ for every $I \in \mathcal{R}(I_1, I_2)$.

Since $t \rightarrow \partial \varphi(t)$ is a maximal monotone multifunction. Each sub-gradient $s_\varphi(\cdot)$ from $\partial \varphi(\cdot)$ is a non-decreasing function. $s_\varphi$ has the following property:

$$\varphi(x - 1(x) - E[X - 1(X)]) - \varphi(x - l^*(x) - E[X - l^*(X)])
\geq s_\varphi(x - l^*(x) - E[X - l^*(X)]) \times (-I(x) + I^*(x)) + E[I(X) - l^*(X)])$$

for all $x \geq 0$. Then, denote $s_\varphi(x)$ by

$$s_\varphi(x - l^*(x) - E[X - l^*(X)])$$

and $s^*(\cdot)$ is a Borel function that satisfies

$$\int_{[0, \infty)} \{ \varphi(x - 1(x) - E[X - 1(X)]) - \varphi(x - l^*(x) - E[X - l^*(X)]) \} dF(x)
\geq \int_{[0, \infty)} s^*(x) \{ -I(x) + l^*(x) \} + E[I(X) - l^*(X)] dF(x).$$

Let $s^*$ be a function supporting $\rho$ at $I^* \in \mathcal{R}(I_1, I_2)$ such that
If $s^*, \lambda \geq 0$ and $I^*[0, \infty) \to (-\infty, \infty)$ are functions such that

(a) for every $x \geq 0$ such that $I^*(x) = l_1(x)$ and

$$
\lambda - s^*(x) + E[s^*(X)] - \lambda \beta \frac{E[I^*(X)]}{\sqrt{Var[I^*(X)]}} + \lambda \beta \frac{l_1(x)}{\sqrt{Var[I^*(X)]}} \geq 0;
$$

(b) for every $x \geq 0$ such that $I^*(x) = l_2(x), l_1(x) < l_2(x)$ and

$$
\lambda - s^*(x) + E[s^*(X)] - \lambda \beta \frac{E[I^*(X)]}{\sqrt{Var[I^*(X)]}} + \lambda \beta \frac{l_1(x)}{\sqrt{Var[I^*(X)]}} \leq 0;
$$

(c) for every $x \geq 0$ such that $l_1(x) < l^*(x) < l_2(x)$ and

$$
\lambda - s^*(x) + E[s^*(X)] - \lambda \beta \frac{E[I^*(X)]}{\sqrt{Var[I^*(X)]}} + \lambda \beta \frac{l_1(x)}{\sqrt{Var[I^*(X)]}} = 0;
$$

(d) $\Pi(I^*(X)) \leq \pi$ and $\lambda(\Pi(I^*(X)) - \pi) = 0$,

then $I^*$ minimizes $\rho(I)$ within the admissible class $\mathcal{R}(l_1, l_2)$ under the restriction $\Pi(I^*(X)) \leq \pi$.

Secondly, they considered the problem under the $L_1$-risk measures and specified the insurer’s risk measure by

$$
\rho_1 = E[I - l(X) - E[I - l(X)]],
$$

and

$$
\rho_1^+ = E[(X - l(X) - E[X - l(X)])^+].
$$

Assume that $\pi, \beta > 0, \pi < E[X] + \beta \sqrt{Var[X]}$ and $\sqrt{Var[Y]} > 0$. There are constants $m_1$ and $m_2$ such that $0 < m_1 < m_2 < \infty$, and $I_1^*(x)$ under $\rho_1$ and $\rho_1^+$ are optimal within class $\mathcal{R}_0$. The optimal contract $I_1^*(x)$ has the following form

$$
I_1^*(X) = \begin{cases} 
0, & x < m_1, \\
X - m_1, & m_1 < x \leq m_2, \\
 - m_1, & x > m_2,
\end{cases}
$$

where $m_1$ and $m_2$ are solutions to

$$
\sqrt{Var[I^*(X)]} > 0.
$$
\[
\begin{align*}
\int_{[0,m_1]} (m_1 - x) dF(x) &= \int_{(m_2,\infty)} (x - m_2) dF(X) \\
\Pi &= E[X] - m_1 + \beta \sqrt{(E[X] - m_1)^2 P(X \leq m_1) + (m_2 - E[X])^2 P(X > m_2)} + \int_{(m_1,m_2)} (x - E[X])^2 dF(x)
\end{align*}
\]

where \( F \) is a distribution function.

Further, they considered the problem under truncated variance risk measure with

\[
\rho_2^+(I) = E[(X - I(X) - E(X - I(X)))^+]^2.
\]

A function \( Q: [0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+ \) is defined by

\[
Q(r, m) = \int_{[0,m]} xdF(x) + \int_{(m,\infty)} (rx + (1-x)m)dF(x).
\]

The optimal contract under the truncated variance risk measure has the following form:

\[
I^*_2(X) = \begin{cases} 
0, & x < m, \\
(1-r)(x - m), & x \geq m,
\end{cases}
\]

where \( m > 0, r \in (0,1) \) are defined by the following equations

\[
\begin{align*}
P &= (1-r) \left[ \int_{(m,\infty)} (x - m)^2 dF(x) + \beta \sqrt{\int_{(m,\infty)} (x - m)^2 dF(x) - \left( \int_{(m,\infty)} (x - m) dF(x) \right)^2} \right] \\
\int_{[0,m]} (m - x) dF(x) - \int_{[0,Q(r,m)]} (Q(r,m) - x) dF(x) &= \frac{r}{\beta} \sqrt{\int_{(m,\infty)} (x - m)^2 dF(x) - \left( \int_{(m,\infty)} (x - m) dF(x) \right)^2}
\end{align*}
\]

5.1.2 Minimize the Underlying Risk with VaR and CVaR

5.1.2.1 Under Expected Value Premium Principles

In Cai et al. (2008), the risk exposure for insurer is defined by

\[
T_i(x) = R_i(X) + \Pi(X).
\]
The ceded function for quota-share reinsurance is \( I(x) = ax \), and the insurer's retained loss function is \( R_I(x) = (1 - a)x \), where \( 0 < a \leq 1 \). In addition, the ceded function and stop-loss reinsurance is
\[
I(x) = (x - d)_+ = \max\{0, x - d\},
\]
and the retained loss function of the insurer is
\[
R_I(x) = \min\{x, d\}, \quad d \geq 0.
\]
Change-loss reinsurance is \( I(x) = a(x - d)_+ \), and
\[
R_I(X) = (1 - a)X + a \cdot \min\{x, d\}.
\]
They use risk measures such as the value-at-risk and the conditional tail expectation with respect to \( X \):
\[
\text{VaR}_\alpha(X) = \inf\{x: \Pr\{X > x\} \leq \alpha\}
\]
and
\[
\text{CTE}_\alpha(X) = \mathbb{E}[X | X \geq \text{VaR}_\alpha(X)].
\]
They used the following notations:
- \( \theta^* = \frac{1}{1 + \theta} \)
- \( d^* = \text{VaR}_{\theta^*}(X) \)
- \( g(x) = (x + \frac{1}{\theta^*}) \int_x^{\infty} S_X(t) dt \)
- \( u(x) = \text{VaR}_X(X) + \frac{1}{\theta^*} \int_{\text{VaR}_X(X)}^{\infty} S_X(t) dt \)

Mathematically, their optimal reinsurance model can be written as follows depending on different risk measure. Under the VaR-optimization, the objective function is
\[
\min_{I \in \mathcal{C}} \{ \text{VaR}_\alpha(T_I(X)) \}
\]
Let \( \mathcal{C} \) denote the class of ceded loss functions which consists of all increasing convex functions \( I(X) \). For a given confidence level \( 1 - \alpha \) with \( 0 < \alpha < S_X(0) \), we have
(a) If \( \theta^* < S_X(0) \) and \( \text{VaR}_\alpha(T_I(X)) > u(\theta^*) \), then the optimal ceded function is attained at \( I^*(x) = (x - d^*)_+ \).
(b) If \( \theta^* < S_X(0) \) and \( \text{VaR}_\alpha(T_I(X)) = u(\theta^*) \), then the optimal ceded function is attained at \( I^*(x) = (x - d^*)_+ \), for any constant \( c \) such that \( 0 < c \leq 1 \).
(c) If $\theta^* \geq S_X(0)$ and $VaR_d(T_1(X)) > g(0)$, then the optimal ceded function is attained at $I^*(x) = x$.
(d) If $\theta^* \geq S_X(0)$ and $VaR_d(T_1(X)) = g(0)$, then the optimal ceded function is attained at $I^*(x) = cx$, for any constant $c$ such that $0 < c \leq 1$.

Under the CTE-optimization, the optimization problem is

$$\min_{t \in \mathbb{C}} \{ CTE_{T_1(x)}(X) \}.$$ 

For a given confidence level $1 - \alpha$, the optimal ceded function are as follows:

(a) If $\alpha < \theta^* < S_X(0)$, and $CTE_{\alpha}(T_1(X)) = u(\theta^*)$, then optimal ceded function is attained at $I^*(x) = (x - d^*)^+$. 
(b) If $\alpha = \theta^* < S_X(0)$ and $CTE_{\alpha}(T_1(X)) = u(\theta^*)$, then optimal ceded function is attained at $I^*(x) = \sum_{j=1}^{n} C_{n,j} (x - d_{n,j})^+$ with $d^* \leq d_{n,1} \leq \cdots \leq \cdots \leq d_{n,n}$ and $n = 1, 2, \ldots$ 
(c) If $\alpha < S_X(0) < \theta^*$, and $CTE_{\alpha}(T_1(X)) = u(\theta^*)$ then the optimal ceded function attained at $I^*(x) = x$.

By formulating an optimization problem that minimizes the VaR (or CTE) of the total cost of the reinsurer, this paper establishes the conditions and examples for optimal reinsure for the risk measure and the safety loading for the reinsurance premium, and used different reinsurance strategies such as stop-loss reinsurance, quota-share reinsurance or change-loss reinsurance in different conditions. And they found that It the conditions for optimal solutions for CTE are less restrictive than those for VaR.

5.1.2.2 Geometric Proof for Cai et al. (2008)

Cheung (2010) reviewed the optimal strategies to minimize Value-at-Risk (VaR) and conditional tail expectation (CTE) under expectation premium principle in Cai et al. (2008) by using geometric arguments. Then he used the same method to solve the VaR-minimization problem under Wang's premium principle as well. He assumed that the survival function, $S(x)$, is strictly decreasing and continuous on $(0, \infty)$, with a possible jump on 0. The ceded loss function $I(x)$ is assumed to be increasing and convex and satisfies

$$0 \leq I(x) \leq x \text{ for } x \geq 0.$$ 

The collection of all possible ceded loss function is denoted as $\mathfrak{C}$. $\mathfrak{B}$ is a subclass of $\mathfrak{C}$ and contains the null function and the linear functions
\[ I_{c,0}(x) = cx, x \geq 0, c \in (0,1]. \]

According to a simple geometric argument, he showed that optimal ceded loss functions must take the form \( I(X) = c(x - d)_+. \) Firstly, he solved the VaR-minimization problem under expectation premium principle. The objective function is

\[
\text{VaR}_\alpha(T(X)) = \text{VaR}_\alpha(R_t(X)) + (1 + \theta) \text{E}[I(X)] \\
= R_t(\text{VaR}_\alpha(X)) + (1 + \theta) \text{E}[I(X)] \\
= \text{VaR}_\alpha(X) - I(\text{VaR}_\alpha(X)) + (1 + \theta) \text{E}[I(X)].
\]

Following Cai et al. (2008), Cheung (2010) used the following notations:

- \( \theta^* = \frac{1}{1 + \theta} \)
- \( d^* = \text{VaR}_{\theta^*}(X) \)
- \( g(x) = (x + \frac{1}{\theta}) \int_0^\infty S_X(t)dt \)
- \( u(x) = \text{VaR}_X(X) + \frac{1}{\theta^*} \int_{\text{VaR}_X(X)}^\infty S_X(t)dt \)

And so, \( g(d^*) = u(\theta^*) \) and \( g(0) = (1 + \theta)\text{E}[X] \) hold. For a given \( \alpha \in (0,S_X(0)) \), the following statements about optimal ceded functions are true to minimize \( \text{VaR}_\alpha(T(X)) \):

(a) If \( \theta^* < S_X(0) \) and \( \text{VaR}_\alpha(X) > u(\theta^*) \), then the minimum value of \( \text{VaR}_\alpha(T(X)) \) over \( \mathbb{C} \) is \( g(d^*) \), and the optimal ceded loss function is \( l^*(x) = (x - d^*)_+ \).

(b) If \( \theta^* < S_X(0) \) and \( \text{VaR}_\alpha(X) > u(\theta^*) \), then the minimum value of \( \text{VaR}_\alpha(T(X)) \) over \( \mathbb{C} \) is \( g(x) \), and the optimal ceded loss function is \( l^*(x) = c(x - d^*)_+ \) for any constant \( c \in [0,1] \).

(c) If \( \theta^* \geq S_X(0) \) and \( \text{VaR}_\alpha(X) > g(0) \), the minimum value of \( \text{VaR}_\alpha(T(X)) \) over \( \mathbb{C} \) is \( g(0) \), the optimal ceded loss function is \( l^*(x) = x \).

(d) If \( \theta^* \geq S_X(0) \) and \( \text{VaR}_\alpha(X) = g(0) \), the minimum value of \( \text{VaR}_\alpha(T(X)) \) over \( \mathbb{C} \) is \( g(0) \), the optimal ceded loss function is \( l^*(x) = cx \) for any constant \( c \in [0,1] \).

(e) For all other cases, the minimum value of \( \text{VaR}_\alpha(T(X)) \) over \( \mathbb{C} \) is \( \text{VaR}_\alpha(X) \), the optimal ceded loss function is \( l^*(x) = 0 \).

Secondly, he solved the CTE-minimization problem under expectation premium principle with the same constraints as the VaR-minimization problem. The objective function is

\[
\text{CTE}_\alpha(T_f(X)) = \text{VaR}_\alpha(T_f(X)) + \frac{E[T_f(X) - \text{VaR}_\alpha(T_f(X))]_+}{P(T_f(X) \geq \text{VaR}_\alpha(T_f(X)))}
\]

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\begin{align*}
&= (T_i(X)) + \frac{E[R_i(X) - VaR_\alpha(R_i(X))]_+}{P(R_i(X) \geq VaR_\alpha(R_i(X)))} \\
&= VaR_\alpha(X) - I(VaR_\alpha(X)) + (1 + \theta)E[I(X)] + \frac{E[(R_i(X) - VaR_\alpha(R_i(X))]_+}{P(R_i(X) \geq VaR_\alpha(R_i(X)))}.
\end{align*}

He came out the following statements for a given $\alpha \in (0, S_X(0))$:

(a) If $\alpha < S_X(0) \leq \theta^*$, then the minimum value of $CTE_\alpha(T_i(X))$ over $\mathcal{B}$ is $g(0)$, and the optimal ceded loss function is $l^*(x) = x$.

(b) If $\alpha < \theta^* < S_X(0)$, then the minimum value of $CTE_\alpha(T_i(x))$ over $\mathcal{B}$ is $g(d^*)$, and the optimal ceded loss function is $l^*(x) = (x - d^*)_+$.

(c) If $\alpha = \theta^* < S_X(0)$, then the minimum value of $CTE_\alpha(T_i(x))$ over $\mathcal{B}$ is $g(VaR_\alpha(X))$, and the optimal ceded loss function can be any function that is identically zero on $[0, VaR_\alpha(X)]$.

(d) If $\alpha > \theta^*$, the minimum value of $CTE_\alpha(T_i(x))$ over $\mathcal{B}$ is $g(VaR_\alpha(X))$, the optimal ceded loss function is the null function $I(X) = 0$.

Finally, he proved the usefulness of his approach through solving the VaR-minimization problem under Wang's premium principle, which is defined by

$$H_w(X) = \int_0^\infty w(S_X(t))dt,$$

where the function $w$ is a non-decreasing, concave function such that $w(0) = 0$ and $w(1) = 1$. The objective function is

$$L(f) = VaR_\alpha(T_i(x))$$

$$= VaR_\alpha(X) - I(VaR_\alpha(X)) + H_w(I(X)).$$

The optimal ceded functions are clarified in the following statements for a given $\alpha \in (0, S_X(0))$:

(a) If $H_w(X) < VaR_\alpha(X)$, then the minimum value of $L$ over $\mathcal{C}$ is $H_w(X)$, and the optimal ceded loss function is $l^*(x) = x$.

(b) If $H_w(X) = VaR_\alpha(X)$, then the minimum value of $L$ over $\mathcal{C}$ is $H_w(X)$, and the optimal ceded loss function is $l^*(x) = cx$ for any constant $c \in (0, 1]$.

(c) If $H_w(X) > VaR_\alpha(X)$, then the minimum value of $L$ over $\mathcal{C}$ is $VaR_\alpha(X)$, and the optimal ceded loss function is $l^*(x) = 0$.

5.1.2.3 Under Variance Related Premium Principles
Chi (2012) derived the optimal ceded function to minimize the Value-at-Risk (VaR) or conditional Value-at-Risk (CVaR) under variance related principles. The ceded and retained loss functions are assumed to be increasing. Guerra and Centeno (2010) first introduced variance related principle

$$\Pi(X) = E[X] + g(\text{var}(X)),$$

where $\text{var}(x)$ is the variance of $X$ and the loading function $g: [0, \infty] \to [0, \infty]$ is increasing with $g(0) = 0$. The layer reinsurance is in the form of

$$\mathcal{L}_{(d,b)}(x) \equiv \min \{(x - d)_+, b - d\} = (x - d)_+ - (x - b)_+, 0 \leq d \leq b \leq \infty$$

where $d$ is the deductible, and $b-d$ is the upper limit of layer reinsurance. The set of admissible ceded loss functions in this paper is

$$\mathcal{C} \equiv \{0 \leq I(x) \leq x: \text{both } R_I(X) \text{ and } I(x) \text{ are increasing functions}\}$$

The optimization problems are

$$\text{VaR}_\alpha(T_I(X)) = \min_{I \in \mathcal{C}} \text{VaR}_\alpha(T_I(X))$$

and

$$\text{CVaR}_\alpha(T_I(X)) = \min_{I \in \mathcal{C}} \text{CVaR}_\alpha(T_I(X)).$$

Assume that $\alpha$ satisfies $0 < \alpha < S_\alpha(0)$. This paper use the following notations: $\underline{x} \equiv \text{ess inf } X$, $x_\alpha \equiv \text{VaR}_\alpha(X)$ and $\overline{x} \equiv \text{ess sup } X$. Define

$$\mathcal{G} \equiv [x, x_\alpha] \times [x_\alpha, \overline{x}]$$

and

$$\mathcal{D} \equiv \left\{ (d, c) \in \mathcal{G}^0: \int_d^c S_X(t) dt = \frac{1}{2\gamma}, c - d = \frac{1}{2\gamma^2} \right\},$$

where $\mathcal{G}$ is the Cartesian product of $[x, x_\alpha]$ and $[x_\alpha, \overline{x}]$, and $\mathcal{G}^0$ represents the interior of $\mathcal{G}$. Define

$$w_1(d) \equiv E \left[ \mathcal{L}_{(d, \text{VaR}_\alpha(X))}(X) \right] = \int_d^{\text{VaR}_\alpha(X)} S_x(t) dt$$

and

$$w_2(d) \equiv E \left[ \mathcal{L}^2_{[d, \text{VaR}_\alpha(X)]}(X) \right] = 2 \int_d^{\text{VaR}_\alpha(X)} (t - d) S_x(t) dt$$

for any $0 \leq d \leq \text{VaR}_\alpha(X)$. In addition, they used the following notations:
\[
\psi(d) \triangleq \frac{w_2(d)}{w_1^2(d)}, \quad 0 \leq d < x_\alpha, \\
\phi(d, c) \triangleq \frac{\int_d^c 2(t - d)S_X(t)dt}{\left(\int_d^c S_X(t)dt\right)^2}, \\
\varphi(d, c) \triangleq \frac{\int_d^c F_X(t)dt}{\sqrt{\phi(d, c) - \int_d^c S_X(t)dt}},
\]
\[
c_v \triangleq \sup \left\{ c \in [x_\alpha, \bar{x}]: \int_x^c F_X(t)dt \leq \frac{(1 - 1)}{2\gamma} \text{ or } c = x_\alpha \right\},
\]
\[
d_s \triangleq \sup \left\{ d \in (x, x_\alpha): E[(X - d)_+] = \frac{1}{2\gamma} \text{ and } \bar{x} - d \leq \frac{1}{2\gamma x_\alpha} \right\}
\]

where \(\sup \emptyset = -\infty\). This paper summarized the optimal ceded loss function under four different situations as below:

(a) Under the assumption of variance premium principle, the ceded loss function to minimize \(\text{VaR}_{\alpha}(T_1(x))\) over \(\mathcal{C}\) is given by
\[
I_\psi^c(x) = \mathcal{L}_{(d_s^*, x_\alpha)}(x)
\]
where
\[
d_s^* \triangleq \min \left\{ x \leq d \leq x_\alpha: E[(X - d)_+] \leq \frac{1}{2\gamma} + E[(X - x_\alpha)_+] \right\}
\]
Moreover, \(I_\psi^c(x)\) is also a solution to minimize \(\text{CVaR}_{\alpha}(T_1(x))\) if \(x_\alpha = \bar{x}\).

(b) Assuming that the survival distribution function \(S_X(t)\) is continuous on \((0, \infty)\) and strictly decreases in a neighborhood of \(x_\alpha\), the ceded loss function to minimize \(\text{CVaR}_{\alpha}(T_1(X))\) with variance premium principle is given by
\[
I_\psi^c(x) = \begin{cases} 
\mathcal{L}_{(d_s^*, c)}(x), & \text{if } |\mathcal{D}| > 0, \\
(x - d_s)_+, & \text{if } |\mathcal{D}| = 0 \text{ and } d_s \neq -\infty, \\
\min\{x, c_v\}, & \text{o.w.}
\end{cases}
\]

(c) When the reinsurance premium is calculated by the standard deviation principle, the ceded loss function \(I_s^c\) to minimize \(\text{VaR}_{\alpha}(T_1(x))\) is given by
\[
I_s^c(x) = \mathcal{L}_{(d_s^*, x_\alpha)}(x),
\]
where
\[
d_s^* \triangleq \min \left\{ \bar{x} \leq d \leq x_\alpha: \psi(d) \geq \theta^2 + 1 \text{ or } d = x_\alpha \right\}
\]
Moreover, \( I^*_s(x) \) is also a solution to the optimal reinsurance to minimize \( CVaR_\alpha(T_I(X)) \) if \( x_\alpha = \bar{x} \).

(d) Define

\[
\mathcal{H} \triangleq \left\{ (d, c) \in g^0 : \frac{c-d}{\int_0^t s_X(t) dt} = \frac{1}{\alpha}, \phi(d, c) = \theta^2 - 1 \right\}.
\]

Assuming that the survival distribution function \( S_X(t) \) is continuous on \((0, \infty)\) and strictly decreases in a neighborhood of \( x_\alpha \), the ceded loss function to minimize \( CVaR_\alpha(T_I(X)) \) with standard deviation premium principle is

\[
I^*_s = \left\{ \begin{array}{ll}
\mathcal{L}_{(d, c)}(x), (d, c) \in \mathcal{H} & \text{if } |\mathcal{H}| > 0, \\
(x - d_s)_+ & \text{if } |\mathcal{H}| = 0 \text{ and } d_s \neq -\infty, \\
0 & \text{if } |\mathcal{H}| = 0, \ d_s = -\infty \text{ and } 1 + \theta^2 > \frac{1}{\alpha}, \\
\min\{x, c_s\} & \text{o.w.}
\end{array} \right.
\]

5.1.2.4 Under Binding and Unbinding Cases

Tan et al. (2011) analyzed the optimal solutions under both binding and unbinding cases depending on the optimal reinsurance premium expenditure relative to the reinsurance premium. The reinsurance model is a constrained optimization model in that one of the constraints can be interpreted as either a reinsurance premium budget or an insurer’s profitability guarantee.

**Expectation principle** is used to determine the premium:

\[
\Pi(I(X)) = (1 + \theta)E[I(X)].
\]

CTE\(_\alpha(T_I)\) is defined as the mean of a random variable with the \( \alpha \)-upper-tail distribution:

\[
\psi_\alpha(\xi) = \begin{cases} 
0, & \text{for } \xi < VaR_\alpha(T_I), \\
\frac{\Pr(T_I \leq \xi) - (1 - \alpha)}{\alpha}, & \text{for } \xi \geq VaR_\alpha(T_I).
\end{cases}
\]

Their goal is find the optimizer for:

\[
\begin{aligned}
\min_{x} \text{CTE}_\alpha(T_I) &= \min_{x} \text{CTE}_\alpha(X - I(X)) + (1 + \theta)E[I(X)] \\
\text{s.t. } 0 \leq I(X) \leq X & \text{ for all } x \geq 0, \\
E[I(X)] & \in \left[0, \frac{\pi}{1+\theta}\right],
\end{aligned}
\]

where \( \alpha, \theta \) and \( \pi \) are constants satisfying
\[0 < \alpha < 1, \theta \geq 0, \text{ and } 0 \leq \pi \leq (1 + \theta)E[X].\]

They assumed that \(X\) has finite first two moments, so the optimal ceded function is restricted to the space

\[\mathcal{L}^2 = \mathcal{L}^2(\Omega, \mathcal{F}, \Pi)\]

with

\[\Omega = (0, \infty],\]

\(\mathcal{F}\) being the Borel \(\sigma\)-field on \(\Omega\).

Let

\[Q = Q_I \cap Q_\pi\]

where

\[Q_I = \{I \in \mathcal{L}^2: 0 \leq I(X) \leq x \text{ for } x \geq 0\}\]

and

\[Q_\pi = \{I \in \mathcal{L}^2: 0 \leq (1 + \theta)E[I] \leq \pi\},\]

respectively.

They discussed it under two cases: unbinding and binding case.

- **Unbinding case**

They defined the mapping \(G_\alpha(\xi, I): \mathbb{R} \times \mathcal{L}^2 \to \mathbb{R}\) such that

\[G_\alpha(\xi, I) = \xi + \frac{1}{\alpha}E[(X - I(X) + (1 + \theta)E[I(X)]) - \xi_+].\]

They found out that minimizing of \(\text{CTE}_\alpha(T_I)\) over \(Q\) is equivalent to minimizing \(G_\alpha(\xi, I)\) over the product space \(\mathbb{R} \times Q\). Then the optimization problem can be written as following:

\[
\begin{cases}
\min_{(\xi, I) \in \mathbb{R} \times Q} G_\alpha(\xi, I) \\
\text{s. t. } E[I(X)] \in \left[0, \frac{\pi}{1+\theta}\right].
\end{cases}
\]

Assume \(\alpha(1 + \theta) \leq 1\), they defined

\[\pi_\alpha = (1 + \theta)E[(X - d_\alpha)_+]\]

where

\[d_\alpha = \inf\{d: \Pr[X > d] \leq \alpha\}\]

and

\[\pi_\theta = (1 + \theta)E[(X - d_\theta)_+],\]

where

\[d_\theta = \inf\{d: \Pr[X > d] \leq \theta\}.
\]

Depending on the level of the reinsurance premium budget, three cases were discussed:
(a) If \( \pi \in (0, \pi_\alpha) \), the \textbf{optimal ceded function} is

\[
I^*(x) = \begin{cases} 
0, & x < \hat{d}, \\
l(x), & x \geq \hat{d}, 
\end{cases}
\]

where the function \( l(x) \) satisfies \( 0 \leq l(x) < x - d_\alpha, \) for \( x \geq \hat{d} \), and the retention \( \hat{d} > 0 \) is determined by \( \mathbb{E}[I^*(X)] = \frac{\pi}{1+\theta} \).

(b) If \( \pi \in [\pi_\alpha, \pi_\theta] \), the \textbf{optimal ceded loss function} to the reinsurance model is of the form:

\[
I^* = (X - d^*)_+,
\]

where \( d^* > 0 \) is solution to

\[
\begin{cases}
(1 + \theta)\mathbb{E}[X - d^*] = \pi, \\
\alpha \leq \mathbb{P}(X \geq d^*) \leq \frac{1}{1 + \theta}.
\end{cases}
\]

(d) If \( \pi \in [\pi_\theta, \infty) \), suppose that \( \alpha(1 + \theta) \leq 1 \) and \( \pi > \pi_\alpha \), then optimal ceded loss function to the reinsurance model is of the form:

\[
I^* = (X - d_\theta)_+.
\]

• **Binding Case:**

The reinsurance model with binding reinsurance premium budget constraint over space \( Q_I \) as follows:

\[
\min_{1 \in Q_I} \text{CTEA}(T_I) \\
\text{s. t. } (1 + \theta)\mathbb{E}[I(X)] = \pi,
\]

which is equivalent to minimizing \( G_\alpha(\xi, 1) \) over the product space \( \mathbb{R} \times Q_I \):

\[
\min_{(\xi, 1) \in \mathbb{R} \times Q_I} G_\alpha(\xi, 1) \\
\text{s. t. } (1 + \theta)\mathbb{E}[I(X)] = \pi.
\]

Assume that \( \alpha(1 + \theta) \leq 1 \), the \textbf{optimal ceded loss function} to the reinsurance problem is

\[
I^*(x) = (x - d^*)_+,
\]

where \( d^* \) is solution to
\[(1 + \theta)E[(X - d^*)_+] = \pi \text{ for each } \pi \in (0, \pi_X]\]

with \(\pi_X = (1 + \theta)E(X)\).

5.1.2.5 Under General Premium Principles

Chi and Tan (2012) analyzed the optimal reinsurance models through minimizing Value at risk (VaR) and conditional value at risk (CVaR) under \textit{general premium principles satisfying three basic axioms: distribution invariance, risk loading and stop-loss ordering preserving}.

Wang's premium principle, expected value principle, and Dutch premium principle are the three representatives of the general premium principles in this paper, which \textit{confines} that both insurer and reinsurer are obligated to pay more for larger loss. The set of admissible ceded loss function is

\[\mathcal{C} \triangleq \{0 \leq l(x) \leq x: \text{both } R_i(X) \text{ and } l(x) \text{ are increasing functions}\}.\]

The layer reinsurance treaty is of the form

\[\min\{(x - d)_+, b\} = (x - d)_+ - (x - (d + b))_+, d, b \geq 0,\]

where \(d\) is the deductible, \(b\) is the upper limit. The risk measures used in this paper are defined by

\[VaR_\alpha(X) \triangleq \inf\{x \geq 0: \mathbb{P}(X > x) \leq \alpha\}, 0 < \alpha < 1,\]

and

\[CVaR_\alpha(X) = \frac{1}{\alpha} \int_0^\alpha VaR_\alpha(X) \, ds, 0 < \alpha < S_X(0).\]

The \textit{objective functions} are

\[VaR_\alpha(T^*_f(X)) = \min_{T_i \in \mathbb{E}} VaR_\alpha(T_i(X)),\]

and

\[CVaR_\alpha(T^*_f(X)) = \min_{T_i \in \mathbb{E}} CVaR_\alpha(T_i(X)).\]

• Wang's premium principle is defined by

\[\Pi_\omega(X) \triangleq (1 + \theta) \int_0^\infty g(S_X(x)) \, dx,\]

where \(\theta \geq 0\) is the safety loading of the reinsurer, and \(g: [0, 1] \rightarrow [0, 1]\) is an increasing and concave function with
\[ g(x) \geq x, \quad g(0) = 0 \text{ and } g(1) = 1. \]

- Expected value premium principle is a special case of the Wang's premium principle, where \( g(x) = x \) and \( \theta > 0 \).

- Dutch premium principle is given by
  \[ \Pi_D(X) \triangleq E[X] + \theta' E[(X - \gamma E[X])_+], \]
  where \( \gamma \geq 1 \) and \( 0 < \theta' \leq 1 \).

The following statements summarizes the optimal ceded functions under different premium principles:

- Under Wang's premium principle, let \( W \) be a random variable satisfying
  \[ S_W(t) = g(S_X(t)), \quad \forall t \in \mathbb{R} \]
  and denote the \( \text{VaR} \) of \( W \) at a confidence level \( \theta/(1 + \theta) \) by
  \[ \beta^* \triangleq \text{VaR}_{\frac{1}{1+\theta}}(W). \]
  The optimal ceded function to minimize \( \text{VaR} \) is
  \[ I^*(x) \triangleq \begin{cases} \min \{ (x - \beta^*)_+, \ \text{VaR}_\alpha(X) - \beta^* \}, & \text{VaR}_\alpha(X) \geq \beta^*; \\ 0, & \text{o.w.} \end{cases} \]
  The optimal ceded function to minimize \( \text{CVaR} \) is
  \[ I^*(x) = \begin{cases} (x - d_1^*)_+ - (x - d_2^*)_+, & \forall d_2 \geq \text{VaR}_\alpha(X), \ \text{VaR}_\alpha(X) = \text{ess sup } X, \\ (x - d_1^*)_+ - (x - d_2^*)_+, & 0. w. \end{cases} \]
  where
  \[ d_1^* \triangleq \min \{ \text{VaR}_\alpha(X), \beta^* \}, \]
  \[ d_2^* \triangleq \min \left\{ \max \left\{ \frac{1}{\alpha(1+\theta)} \left( \frac{1}{\alpha(1+\theta)} \right), \text{VaR}_\alpha(X) \right\}, \text{ess sup } X \right\}. \]
  If \( 1 + \theta \geq \frac{1}{\alpha} \), \( I^*(x) = 0 \).
• Under the expected value premium principle, the optimal ceded function to minimize VaR is

\[ I^*(x) \triangleq \begin{cases} (x - \text{VaR}_{1+\theta}(X))_+ - (x - \text{VaR}_\alpha(X))_+, & \text{VaR}_\alpha(X) \geq \text{VaR}_{1+\theta}(X); \\ 0, & \text{o.w.} \end{cases} \]

• Under the Dutch premium principle, the optimal ceded function to minimize VaR is

\[ I^*(x) = \min \{x, \text{VaR}_\alpha(X)\}; \]

the optimal ceded function to minimize CVaR is

\[ I^*(x) = \begin{cases} \min \{x, \mathcal{U}^*\}, & 1 + \theta' > \frac{1}{\alpha} \text{ and } c_0 < \text{ess sup} X, \\ x, & \text{o.w.} \end{cases} \]

where

\[ \mathcal{U}^* \triangleq \min \left\{ \max \left\{ m^{-1} \left( \frac{1/\alpha - (1 + \theta)}{\theta' - \theta} \right), c_0 \right\}, \text{ess sup} X \right\}. \]

5.1.2.6 Arrangements in the Presence of Two Reinsurers

Chi and Meng (2014) studied optimal reinsurance concerning arrangements in the presence of two reinsurers. Since reinsurers may have different risk tolerances, the insurer could pay less cost for ceding an amount of loss by formulating a competitive reinsurance portfolio. Based on that, their objective is to minimize total risk under VaR and CVaR risk measures.

They first assume that the premium principle faced by first insurer is expected value principle

\[ \pi_1(\cdot) = (1 + \theta)E[\cdot], \]

while the premium principle \( \pi_2(\cdot) \) faced by second insurer satisfies three axioms: distribution invariance, risk loading, and preserving stop-loss order. In order to exclude the moral hazard, the set of admissible ceded strategies is defined by

\[ \mathcal{C} := \{I \in \mathbb{R}_+^2: \text{both } R_1(X) \text{ and } I_{\text{sum}}(X) \text{ are increasing with } 0 \leq R_1(X) \leq X\}. \]

Let \( I_1(X) \) be the amount of loss ceded to the first reinsurer and \( I_2(X) \) be the amount of loss ceded to the second reinsurer. Thus, the ceded loss function is \( I_{\text{sum}}(X) = I_1(X) + I_2(X) \). And the residual loss function is \( R_1(X) = X - I_{\text{sum}}(X) \).
They assume \( \pi_1 \) and \( \pi_2 \) are premium principles. Then, the total risk exposure of the insurer is

\[ T_1(X) := R_1(X) + \pi_1(I_1(X)) + \pi_2(I_2(X)). \]

The optimization problems considered in this paper are

\[ \text{VaR}_\alpha(T_1(X)) = \min_{t \in \mathcal{C}} \text{VaR}_\alpha(T_1(X)) \]

and

\[ \text{CVaR}_\alpha(T_1(X)) = \min_{t \in \mathcal{C}} \text{CVaR}_\alpha(T_1(X)). \]

In conclusion, an optimal reinsurance strategy for an insurer is to cede two adjacent layers over both the VaR and CVaR criteria, where the upper layer is assigned to the reinsurer with expected value premium principle.

Furthermore, they also derive explicitly the optimal reinsurance by assuming that \( \pi_2 \) follows the generalized Wang's premium principle, which is formally defined by

\[ \pi_2(X) = (1 + \varphi) \int_0^\infty (S_X(x)) dx, \]

where \( \varphi \geq 0 \) represents the administrative or other expenses related to such a provision of reinsurance. Note that \( g(S_X(t)) \) is a survival distribution function of a nonnegative random variable. Without loss of generality, this random variable is denoted by \( W \), i.e.

\[ S_W(t) = g(S_X(t)), \forall t \in \mathbb{R}_+. \]

They then define

\[ K(t) = \frac{S_W(t)}{S_X(t)}, t < \text{ess sup } X, \]

and then a generalized inverse function of \( K(t) \) is defined by
Define a layer \((a, b)\) of a given risk \(X\) is defined by

\[
\ell_{a,b}(X) := \min\{(X - a)_+, b - a\}, \quad 0 \leq a \leq b.
\]

1) The optimal ceded strategy for the VaR-based reinsurance model is

\[
I^*(x) = \begin{cases} 
(0, \ell_{\beta, \text{VaR}_\alpha(X)}(x)), & \gamma \geq \text{VaR}_\alpha(X), \\
(\ell_{\xi, \text{VaR}_\alpha(X)}(x), 0), & \gamma = 0 \text{ or } 0 < \gamma < \beta, \\
(\ell_{\gamma, \text{VaR}_\alpha(X)}(x), \ell_{\beta, \gamma}(x)), & \text{o.w.}
\end{cases}
\]

where \(\beta = \text{VaR}_{\frac{1}{1+\theta}}(W), \xi = \text{VaR}_{\frac{1}{1+\theta}}(X)\) and \(\gamma = K^{-1}\left(\frac{1+\theta}{1+\varphi}\right)\).

2) The optimal ceded strategy for the CVaR-based reinsurance model:

\[
I^*(x) = \begin{cases} 
((x - \xi)_+, 0), & \gamma < \beta, \\
((x - \gamma)_+, \ell_{\beta, \gamma}(x)), & \text{o.w.}
\end{cases}
\]

where \(\beta, \xi\) and \(\gamma\) are defined same as in 1).

For the classical Wang's premium principle, \(\varphi = 0\) such that \(\beta = 0\), then the above proposition implies that under the criteria of CVaR risk measure, the optimal strategy for an insurer is to cede all the loss to two reinsurers, i.e. \(I^*(x) = ((x - \gamma)_+, x \wedge y)\).

5.1.2.7 Optimal Problem with Regulatory Initial Capital and Counterparty Default Risk

Cai et al. (2014) investigated the situation that the reinsurer fails to pay the promised amount when the promised amount exceeds the reinsurer's solvency. In the presence of the regulatory initial capital and the counterparty default risk, they researched on the optimal reinsurance designs from an insurer's point of view and derive optimal reinsurance strategies that maximize the expected utility of an insurer's terminal wealth or minimize the VaR of an insurer's total retained risk.

They first assume that the initial capital or reserve of a reinsurer in a reinsurance contract \(I\) is determined through regulation by VaR of its promised indemnity \(I(X)\), and denote the initial capital of the reinsurer by

\[
\omega_1 = \text{VaR}_\alpha(I(X)),
\]
where \( \text{VaR}_\alpha(Z) = \inf\{z : \Pr[Z \geq z] \leq \alpha\} \) is the VaR of a random variable \( Z \), and \( 0 < \alpha < 1 \) is called the risk level. And a feasible reinsurance contract \( I \) should satisfy the following two conditions:

- \( I : [0, \infty) \rightarrow [0, \infty) \) such that \( I(0) = 0 \) and \( I \) is non-decreasing
- \( I(y) - I(x) \leq y - x \) for any \( 0 \leq x \leq y \)

Denote the set of all feasible reinsurance contracts satisfying conditions 1) and 2) by \( J \).

They also assume that the reinsurer charges a reinsurance premium \( P_I \) based on the promised indemnity \( I(X) \). In the first part of the paper, the objective is to maximizing the expected utility of its terminal wealth of under an increasing concave utility function \( u \):

\[
\max_{I \in \mathcal{C}} E[u(\omega_0 - X + I(X) \wedge (\omega_I + P_I) - P_I)]
\]

such that \( P_I = (1 + \theta)E[I(X)] = p \), where \( 0 < p \leq (1 + \theta)E[X] \) with \( \theta > 0 \) is a given reinsurance premium budget for the insurer.

In the second part of the paper, they assume that the insurer wants to use VaR at a risk level \( 0 < \beta < 1 \) to control its total retained risk and then seeks an optimal reinsurance strategy \( I^* \) that minimizes this VaR:

\[
\min_{I \in \mathcal{C}} \text{VaR}_\beta(X - I(X) \wedge (\omega_I + P_I) + P_I).
\]

To solve the first objective, denote \( a = \text{VaR}_\alpha(X) \), \( H(I) = E[u(X - I(X) \wedge (\omega_I + P_I))] \). For any \( I \in \mathcal{C}_p \), there exist a \( k_I \in \mathcal{C}_p \) such that \( H(k_I) \leq H(I) \) and \( k_I \) is defined as

\[
k_I(x) = \begin{cases} 
(x - d_1)^+ & \text{for } 0 \leq x < d_1, \\
x - d_1 & \text{for } d_1 \leq x < d_1 + I(a), \\
I(a) & \text{for } d_1 + I(a) \leq x < d_2, \\
I(a) + x - d_2 & \text{for } d_2 \leq x < d_2 + p, \\
I(a) + p & \text{for } d_2 + p \leq x < d_3, \\
I(a) + p + x - d_3 & \text{for } d_3 \leq x.
\end{cases}
\]

for some \((d_1, d_2, d_3)\) satisfies \( 0 \leq d_1 \leq d_1 + I(a) \leq a \leq d_2 \leq d_2 + p \leq d_3 \leq \infty \).

Assume \( 0 < p < (1 + \theta)E[X] \), denote the optimal solution to above equations by \( I^* \).

Furthermore, define contract \( I_{M, \xi} \in \mathcal{C} \) as
\[ I_{M,ξ} = x - (x - ξ)^+ + (x - a)^+ = \begin{cases} 
\xi, & \text{for } 0 \leq x < ξ, \\
x, & \text{for } ξ \leq x < a, \\
ξ + x - a, & \text{for } a \leq x < a + p. 
\end{cases} \]

and denote the reinsurance premium based on \( I_{M,ξ} \) by \( p_{M,ξ} \). In order to identify all the valid \( ξ \) so that \( I \in ℭ_{p,ξ} : I(a) = ξ \neq \emptyset \), we also denote

\[ ξ_0 = \inf \{ ξ \in [0, a] \text{ such that } p_{M,ξ} \geq p \} \]

and

\[ ξ_M = \sup \{ ξ \in [0, a] \text{ such that } p_{0,ξ} \leq p \}. \]

Then, for a given \( ξ \in [ξ_0, ξ_M] \),

\[
I^*(x) = (x - d_1^*)^+ - (x - a)^* + (x - d_2^*)^+ - (x - d_2^* - p)^* + (x - d_3^*)^* \\
= \begin{cases} 
0, & \text{for } 0 \leq x < d_1^*, \\
x - d_1^*, & \text{for } d_1^* \leq x < a, \\
a - d_1^*, & \text{for } a \leq x < d_2^*, \\
a - d_1^* + x - d_2^*, & \text{for } d_2^* \leq x < d_2^* + p, \\
a - d_1^* + p, & \text{for } d_2^* + p \leq x < d_3^*, \\
a - d_1^* + p + x - d_3^*, & \text{for } d_3^* \leq x,
\end{cases}
\]

where

\[
(d_1^*, d_2^*, d_3^*) = \begin{cases} 
(0, a, d_3,a), & \text{if } ξ_1 = a, \\
(0, d_2,a, +∞), & \text{if } ξ_1 \leq a \text{ and } h'(ξ_M) \leq 0, \\
(a - ξ^*, d_2,ξ^*, +∞), & \text{if } ξ_1 \leq a \text{ and } h'(ξ_M) > 0
\end{cases}
\]

with \( d_{2,ξ} \) determined by

\[ (1 + θ)E[I^*_ξ(X)] = p, \]

and \( d_{2,a}, d_{3,a} \) determined by

\[ (1 + θ)E[I^*_ξ(X)] = p \text{ and } h(ξ) = H(I^*_ξ). \]

Then, they are trying to solve the second objective. For any \( I \in ℋ \), denote

\[ V(I) = \text{VaR}_β(X - I(X) \wedge (ω_I + p_I)) + P_I. \]

Again, \( a = \text{VaR}_α(X) \) and \( b = \text{VaR}_β(X) \). For any \( I \in ℋ \), there exist a \( m_I \in ℋ \) satisfying \( V(m_I) \leq V(I) \) and \( m_I \) is defined as
\[ m_I(x) = (x - d_1)^+ - (x - a \land b)^+ + (x - d_2)^+ - (x - a \lor b)^+ \]

\[
\begin{cases} 
0, & \text{for } 0 \leq x < d_1, \\
x - d_1, & \text{for } d_1 \leq x < a \land b, \\
a \land b - d_1, & \text{for } a \land b \leq x < d_2, \\
a \land b - d_1 + x - d_2, & \text{for } d_2 \leq x < a \lor b, \\
a + b - d_1 - d_2, & \text{for } a \lor b \leq x < \infty,
\end{cases}
\]

where

\[ d_1 = a \land b - I(a \land b) \]

and

\[ d_2 = a \lor b - (I(a \lor b) - I(a \land b)) \]

satisfying

\[ 0 \leq d_1 \leq a \land b \leq d_2 \leq a \lor b. \]

Then, the optimal solution is

\[ I^*(x) = (x - d^*)^+ - (x - b)^+, \]

where

\[ d^* = b \land VaR_{\frac{1}{1+\theta}}(X) \text{ if } \alpha \leq \beta, \]

and

\[ d^* = \max \left\{ 0, d_0 \land VaR_{\frac{1}{1+\theta}}(X) \right\} \text{ if } \alpha > \beta \text{ and } \alpha \leq \frac{1}{1+\theta}. \]

Based on the first assumption, they derived the optimal contract under the absolute deviation risk:

\[ I_1^*(X) = \begin{cases} 
0, & x < m_1, \\
x - m_1, & m_1 < x \leq m_2, \\
m_2 - m_1, & x > m_2,
\end{cases} \]

where \( R_1^* \) is a stop-loss contract with a contractual maximum payment. The constants \( m_1 \) and \( m_2 \) are defined by the following equations:

\[
\int_{[0,m_1]} (m_1 - x)dF(x) = \int_{(m_2,\infty)} (x - m_2)dF(X),
\]
where \( F \) is any distribution function, and

\[
\Pi = E[X] - m_1 + \beta \sqrt{(E[X] - m_1)^2 P(X \leq m_1) + (m_2 - E[X])^2 P(X > m_2) + \int_{(m_1, m_2)} (x - E[X])^2 dF(x)}.
\]

Under the second assumption, a function \( Q: [0, 1] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) was defined as follows:

\[
Q(r, m) = \int_{[0,m]} x dF(x) + \int_{(m, \infty)} (rx + (1 - x)m) dF(x).
\]

The optimal contract under the truncated variance risk measure has the following form:

\[
I^*_2(X) = \begin{cases} 
0, & x < m, \\
(1 - r)(x - m), & x \geq m,
\end{cases}
\]

where \( m > 0, r \in (0, 1) \). Note that \( r \) and \( m \) are defined by the following equations,

\[
P = (1 - r) \int_{(m, \infty)} (x - m)^2 dF(x) + \beta \sqrt{\int_{(m, \infty)} (x - m)^2 dF(x) - \left( \int_{(m, \infty)} (x - m) dF(x) \right)^2}
\]

and

\[
\int_{[0,m]} (m - x) dF(x) - \int_{[0,Q(r,m)]} (Q(r, m) - x) dF(x)
\]

\[
= \frac{r}{\beta} \sqrt{\int_{(m, \infty)} (x - m)^2 dF(x) - \left( \int_{(m, \infty)} (x - m) dF(x) \right)^2}
\]

where \( I^*_2 \) is a change loss arrangement with suitably chosen change loss point \( m \) and quota share coefficient \( r \) within the class \( \mathcal{R}_0 \), where \( \mathcal{R}_0 \) is defined by the boundary functions

\[
I_1(x) \equiv 0, I_2(x) \equiv 0.
\]

5.1.3 Under Ruin Probability

5.1.3.1 Insurer with Dependent Business Lines

In real life, insurers usually have many lines of business, such as auto insurance/life insurance. Therefore, Bai et al. (2013) discussed the case when an insurer has two dependent lines of business. They assumed that the insurer applies a reinsurance policy to each line of business to protect the business from large losses and to reduce the ruin probability of the insurer and they
modeled the reserves of the two lines of business by a two-dimensional compound Poisson risk process or a common shock model. In their model, claim sizes \( \{X_i, i = 1, 2, \ldots \}, \{Y_i, i = 1, 2, \ldots \} \) are i.i.d. positive random variables with common distribution function \( F_X(\cdot), (F_Y(\cdot)) \), mean value \( u_X(u_Y) \).

Note that \( \sum_{i=1}^{M_1(t)} I_1(X_i), \sum_{i=1}^{M_2(t)} I_2(Y_i) \) are the aggregated claims up to time \( t \) in the two lines of business, respectively. And \( C_1(t) \) and \( C_2(t) \) are accumulated premiums up to time \( t \) in the two lines of business, respectively, which are determined by the expected value principle with positive safety loading \( \eta_i, i = 1, 2 \), namely,

\[
\begin{align*}
C_1 &= (1 + \eta_1)(\lambda_1 + \lambda)u_X, \\
C_2 &= (1 + \eta_2)(\lambda_2 + \lambda)u_Y.
\end{align*}
\]

And \( I_1 \) and \( I_2 \) are reinsurance strategies to lines 1 and 2. Reinsurance premium \( \Pi_1^\pi(t), \Pi_2^\pi(t) \) are calculated by the expected value principle:

\[
\begin{align*}
\Pi_1^\pi(t) &= (\lambda_1 + \lambda)[(1 + \theta_1)E[I_1(X_i)] - (\theta_1 - \eta_1)u_X], \\
\Pi_2^\pi(t) &= (\lambda_2 + \lambda)[(1 + \theta_2)E[I_2(X_i)] - (\theta_2 - \eta_2)u_Y].
\end{align*}
\]

Moreover, \( M_1(t) \) and \( M_2(t) \) are defined by

\[
M_1(t) = N_1(t) + N(t)
\]

and

\[
M_2(t) = N_2(t) + N(t),
\]

where \( \{N_1(t)\}, \{N_2(t)\}, \{N(t)\} \) are independent Poisson processes. Thus, the total reserve of the insurance portfolio up to time \( t \) under a two-dimensional reinsurance policy, denoted by \( \{R(t), t \geq 0\} \), is

\[
R^\pi(t) = u + [C_1(t) - \Pi_1^\pi(t)] + [C_2(t) - \Pi_2^\pi(t)] - \sum_{i=1}^{M_1(t)} I_1(X_i) - \sum_{i=1}^{M_2(t)} I_2(Y_i).
\]

Their goal is to obtain the value function defined by

\[
\varphi(X) = \min \varphi_\Pi(X),
\]

where

\[
\varphi_\Pi(X) = \Pr \{\tau_\Pi < \infty | R^\pi(0) = x\}
\]

and

\[
\tau_\Pi = \inf \{t \geq 0: R^\pi(t) < 0\}.
\]

and to find an optimal strategy
\[ I^* = (d_1^*, d_2^*) \in \Pi. \]

If \( \varphi(X) \) is twice continuously differentiable on \((0, \infty)\), then \( \varphi(X) \) satisfies the following Hamilton-Jacobi-Bellman equation:

\[
\min_{0 \leq d_i \leq \infty, i=1,2} \{ (\lambda + \lambda_1)[\theta_1 g_X(d_1) + (\eta_1 - \theta_1)u_X] + \\
(\lambda + \lambda_2)[\theta_2 g_Y(d_2) + (\eta_2 - \theta_2)u_Y] \varphi'(x) + \\
\frac{1}{2} [ (\lambda + \lambda_1)G_X(d_1) + (\lambda + \lambda_2)G_Y(d_2) + 2\lambda g_X(d_1)g_Y(d_2)] \varphi''(x) = 0
\]

with boundary conditions:

\[ \varphi(0) = 1 \text{ and } \varphi(\infty) = 1. \]

Assume that \( v(x) \) is a twice continuously differentiable solution. Then, \( d_1^*(x) \) and \( d_2^*(x) \) satisfy

\[
(\lambda + \lambda_1)[\theta_1 g_X(d_1^*(x)) + (\eta_1 - \theta_1)u_X] + (\lambda + \lambda_2)[\theta_2 g_Y(d_2^*(x)) + (\eta_2 - \theta_2)u_Y] \varphi'(x) + \\
\frac{1}{2} [ (\lambda + \lambda_1)G_X(d_1^*(x)) + (\lambda + \lambda_2)G_Y(d_2^*(x)) + 2\lambda g_X(d_1^*(x))g_Y(d_2^*(x))] \varphi''(x) = 0
\]

for all \( x > 0 \).

Moreover,

\[
l_v(d_1, d_2) = (\lambda + \lambda_1)[\theta_1 g_X(d_1) + (\eta_1 - \theta_1)u_X]v'(x) + \\
(\lambda + \lambda_2)[\theta_2 g_Y(d_2) + (\eta_2 - \theta_2)u_Y]v'(x) + \frac{1}{2} [ (\lambda + \lambda_1)G_X(d_1) + \\
(\lambda + \lambda_1)G_X(d_2) + 2\lambda g_X(d_1)g_Y(d_2)]v''(x),
\]

where

\[
g_X(d) = E[X_i \wedge d], \\
g_Y(d) = E[Y_i \wedge d], \\
G_X(d) = E[X_i \wedge d]^2,
\]

and

\[
G_Y(d) = E[Y_i \wedge d]^2.
\]

By differentiating \( l_v(d_1, d_2) \) with respect to \( d_1 \) and \( d_2 \), \( d_1^*(x), d_2^*(x) \) are solutions to

\[
\begin{align*}
(\lambda + \lambda_1)\theta_1 v'(x) + [(\lambda + \lambda_1)d_1^* + \lambda g_Y(d_2^*)]v''(x) &= 0, \\
(\lambda + \lambda_2)\theta_2 v'(x) + [(\lambda + \lambda_2)d_2^* + \lambda g_Y(d_1^*)]v''(x) &= 0.
\end{align*}
\]

Define
\[ l_x(x) = \theta_2 x - \frac{\lambda \theta_1}{\lambda + \lambda_2} g_{X}(x), \]
\[ l_y(x) = \theta_1 x - \frac{\lambda \theta_2}{\lambda + \lambda_1} g_{Y}(x), \]
and
\[ k(x) = (\lambda + \lambda_1) \theta_1 \left[ g_{X}(x) - \frac{g_{X}(x)}{2x} \right] + k_0, \]
where
\[ k_0 = (\lambda + \lambda_1) (\eta_1 - \theta_1) u_x + (\lambda + \lambda_2) (\eta_2 - \theta_2) u_y < 0. \]

Hence, \( d_1^*(x) \) and \( d_2^*(x) \) are solutions to
\[
\begin{cases}
(\lambda + \lambda_1)[\theta_1 g_{X}(d_1^*(x)) + (\eta_1 - \theta_1) u_x] + (\lambda + \lambda_2)[\theta_2 g_{Y}(l_y^{-1}(d_1^*(x))) + (\eta_2 - \theta_2) u_y] \\
\theta_1 [(\lambda + \lambda_1) g_{X}(d_1^*(x)) + (\lambda + \lambda_2) g_{Y}(l_y^{-1}(d_1^*(x))) + 2 \lambda g_{X}(d_1^*(x)) g_{Y}(l_y^{-1}(d_1^*(x)))]
\end{cases}
\]
\[ d_1^*(x) + \frac{\lambda}{\lambda + \lambda_1} g_{Y} \left( l_y^{-1}(d_1^*(x)) \right) = 0, \]
\[ d_2^* = l_y^{-1}(d_1^*(x)). \]

5.1.4 Minimizing the General Risk with Distortion Premium

Assa (2014) tried to find the optimal ceded loss functions that could solve the unification of ceding problem, reinsurance problem and social planner problem. The reinsurance problem is to minimize the total risk of reinsurer. Social planner problem is to minimize the economy's total risk.

Let \( \Pi : [0, 1] \rightarrow [0, 1] \) be a non-decreasing function such that \( \Pi(0) = \Pi(1) - 1 = 0 \). The distortion premium \( \pi_\Pi \) is defined as
\[ \pi_\Pi(X) = \int_0^1 \text{VaR}_t(X) d \Pi(t). \]

Define
\[ \mathcal{C} = \{ 0 \leq I(x) \leq x | I(x) \text{ and } x - I(x) \text{ are nondecreasing} \}. \]

His objective function is
\[ \min_{I \in \mathcal{C}} \{ a_1 \Lambda_1(X - I(X)) + a_2 \Lambda_2(I(X)) \}, \]
where \( \Lambda_1, \Lambda_2 \) are either distortion risk measure or premium and \( a_1, a_2 \) are positive numbers. He introduced that
\[ \Psi = (a_2 - a_1) - (a_2 \Pi_2(t) - a_1 \Pi_1(t)), \]

\[ h^*(t) = \begin{cases} 0, & \text{if } \Psi(t) > 0, \\ 1, & \text{if } \Psi(t) < 0, \\ \bar{h}(t), & \text{o.w.} \end{cases} \]

where \( \bar{h} \) could be any function between 0 and 1 on \( \Psi = 0 \).

If \( \Lambda_1, \Lambda_2 \) satisfy \( \lim_{n \to \infty} \Lambda_i(X \land n) = \Lambda_i(X), \quad i = 1,2, \) the optimal ceded loss function is

\[ l^*(X) = \int_0^X h^*(VAR_t(X))dt. \]

### 5.1.5 Under Expectile Risk Measure

Cai and Weng (2016) published a paper concerning the liability of the insurer. The framework in this paper of optimal reinsurance is from the insurer's perspective rather than the reinsurer's point of view. They are trying to minimize the liability of an insurer, which is defined as the actuarial reserve on an insurer's risk exposure plus the risk margin required for the risk exposure.

Let \( T_i(x) \) be risk exposure of the insurer \( L_i(X) \) be the liability of the insurer and define \( E(Z; \omega) \) as the expectile of a loss random variable \( Z \) with \( E[Z^2] < \infty \) at a confidence level \( \omega \).

\[ E(Z; \omega) = \arg \min_{m \in \mathbb{R}} \{ \omega E[(Z - m)^+_{\omega}] + (1 - \omega) E[(m - Z)^+_{\omega}] \}, \]

where \( (x)_+ = \max(x,0) \).

Also, in order to preclude moral hazard in a reinsurance contract, both the ceded loss function \( l(X) \) and retained loss function \( R_i(X) \) are assumed to be increasing and continuous, that is,

\[ \mathcal{C} = \{ 0 \leq l(x) \leq x; \text{both } R_i(x) \text{ and } l(x) \text{ are increasing in } x \}. \]

If using the expectile to calculate the risk margin, the liability of insurer, denoted by \( L_i(X) \), is

\[ L_i(X) = E[T_i(X)] + \delta \cdot E(T_i(X)) - E[T_i(X)]; \ \omega \]

And the objective function be

\[ \begin{cases} \min_{l \in \mathcal{C}} L_i(X) \\ s.t. \ \Pi(l(X)) \leq \pi \end{cases} \]

where \( \pi > 0 \) is constant representing the insurer’s reinsurance premium budget.

Also, in the paper, they consider premium principles satisfying following three conditions:
C1 (Law invariance): $\Pi(Y)$ depends only on the distributional law $I_Y(\cdot)$ of $Y$;

C2 (Risk-loading property): $\Pi(Y) \geq E[Y]$ for any loss random variable $Y$;

C3 (Preserving the convex order): $\Pi(Y) \leq \Pi(Z)$ for any loss random variables $Y$ and $Z$ with $Y \leq_{cx} Z$, i.e. $E[h(Y)] \leq E[h(Z)]$ for all convex function $h$, for which the corresponding expectations exist.

They explore the application of the risk measure of expectiles in optimal reinsurance designs and identify the optimal reinsurance forms that minimize the liability of the insurer. Note that the set of premium principles is defined by

$$\mathcal{P} = \{\text{Premium principle } \Pi : \Pi \text{ satisfies C1, C2, C3}\}.$$

They reach two major conclusions.

1) For any $\Pi \in \mathcal{P}$, they proved that a two-layer reinsurance treaty is optimal.

Define a two-layer ceded loss function $h(x)$, for $0 \leq a \leq b \leq m \leq \infty$,

$$h(x) = h_{a,b,m}(x) = x - (x - a)_+ + (x - b)_+ - (x - m)_+$$

$$= \begin{cases} 
  x, & 0 \leq x < a, \\
  a, & a \leq x < b, \\
  x - (b - a), & b \leq x < m, \\
  m - (b - a), & x \geq m,
\end{cases}$$

Consider $E[(X - m)_+] = E[(R_{h_{a,b,m}}(X) - \xi_{a,b,m}^R)^+]$ with $\xi_{a,b,m}^R$ denoting the expectile of $R_{h_{a,b,m}}(X) = X - h_{a,b,m}(X)$ and define

$$\mathcal{C}_0 = \{h_{a,b,m} : 0 \leq a \leq b = a + \xi_{a,b,m}^R \leq m \leq \infty, \text{ and } m \text{ satisfies } E[(X - m)_+]\}.$$ 

For $\Pi \in \mathcal{P}$ and reinsurance premium budget $\pi > 0$, denote

$$\mathcal{G} = \{I \in \mathcal{C} : \Pi(I(X)) \leq \pi\}$$

and $\mathcal{G}_0 = \{I \in \mathcal{C}_0 : \Pi(I(X)) \leq \pi\}$.

They found

$$\min_{I \in \mathcal{G}_0} L_I(X) = \min_{h_{a,b,m} \in \mathcal{G}_0} L_{h_{a,b,m}}(X)$$

and
\[ L_{h,a,b,m}(X) = E[T_{h,a,b,m}(X)] - \delta \left( E_{h,a,b,m}^T - E[T_{h,a,b,m}(X)] \right) \]

\[ \leq E[T_1(X)] + \delta (E_1^T - E[T_1(X)]) \]

\[ = L_1(X), \]

where \( E_1^T = E(T_1(x); \alpha) \).

2) Under the expected value premium principle, they show that a one-layer treaty is optimal.

Define a one-layer ceded loss function for \( 0 \leq d \leq m \leq \infty \),

\[ I_{d,m}(x) = (x - d)_+ - (x - m)_+ = \begin{cases} 0, & 0 \leq x < d, \\ x - d, & d \leq x < m, \\ m - d, & x \geq m. \end{cases} \]

And

\[ \mathcal{C}_1 = \left\{ I_{d,m} \in \mathcal{C}_0 : 0 \leq \mathcal{E}_{id,m}^R = d \leq m \leq \infty \right\}. \]

Again, they found

\[ \min_{I \in \mathcal{G}} L_I(X) = \min_{I_{d,m} \in \mathcal{G}_1} L_{I_{d,m}}(X) \]

and

\[ L_{I_{d,m}}(X) = E[T_{I_{d,m}}(X)] - \delta \left( E_{I_{d,m}}^T - E[T_{I_{d,m}}(X)] \right) \]

\[ \leq E[T_{h,a,b,m}(X)] - \delta \left( E_{h,a,b,m}^T - E[T_{h,a,b,m}(X)] \right) \]

\[ = L_{h,a,b,m}(X). \]

### 5.2 Maximizing Expected Utility of Wealth

Liang and Yuen (2014) studied optimal dynamic reinsurance with dependent risks. Their objective is to consider the optimal proportional reinsurance strategy in a risk model with two dependent classes of insurance business, where the two claim number processes are correlated through a common shock component. The insurer is interested in maximizing the expected utility of terminal wealth.

They assume that the insurer has an exponential utility function,
\[ u(x) = -\frac{m}{A} e^{-Ax}, \]

where \( A \) is risk aversion parameter and \( m > 0 \).

Let \( X_i \) be the claim size random variables for the first class with common distribution \( F_X(x) \), and \( Y_i \) be the claim size random variables for the second class with \( F_Y(y) \). \( X \) and \( Y \) are independent.

Assume that

\[ F_X(x) = 0 \text{ for } x \leq 0, \quad F_Y(y) = 0 \text{ for } y \leq 0 \]

and

\[ 0 < F_X(x) < 1 \text{ for } x > 0, \quad 0 < F_Y(y) < 1 \text{ for } y > 0. \]

The two claim number processes are correlated in the way that

\[ M_1(t) = N_1(t) + N(t) \]

and

\[ M_2(t) = N_2(t) + N(t), \]

where \( N_1(t), N_2(t) \) and \( N(t) \) are three independent Poisson processes with parameters \( \lambda_1, \lambda_2 \) and \( \lambda_3 \). Therefore, the aggregate claims process generated from the two classes of business has the form

\[ S_t = \sum_{i=1}^{N_1(t) + N(t)} X_i + \sum_{i=1}^{N_2(t) + N(t)} Y_i. \]

If the insurance company is allowed to continuously reinsure a fraction of its claim with the retention levels \( q_{1t} \in [0,1] \) and \( q_{2t} \in [0,1] \) for \( X_i \) and \( Y_i \) respectively. The company is also allowed to invest all its surplus in a risk-free interest rate \( r \). Let the reinsurance premium rate at be \( \delta(q_{1t}, q_{2t}) \), let \( \Psi_t^{q_{1t}q_{2t}} \) be the wealth of the insurer at time \( t \) under the strategy \((q_{1t}, q_{2t})\). Then, the above process evolves as

\[ d\Psi_t^{q_{1t}q_{2t}} = \left[r \Psi_t^{q_{1t}q_{2t}} + (c - \delta(q_{1t}, q_{2t}))\right]dt - q_{1t}dS_1(t) - q_{2t}dS_2(t) \]

and

\[ d\Phi_t^{q_{1t}q_{2t}} = \left[r \Phi_t^{q_{1t}q_{2t}} + (c - \delta(q_{1t}, q_{2t}))\right]dt - q_{1t}a_1 - q_{2t}a_2, \]

where \( a_1 = (\lambda_1 + \lambda)E(X) \) and \( a_2 = (\lambda_2 + \lambda)E(Y) \).

Assume that the insurer is interested in maximizing the expected utility of terminal wealth at time \( T \), which is the **objective function** defined by
In the compound Poisson risk model, the optimal reinsurance strategy for the aggregate claims process is

\[
J^{q_1q_2}(t, x) = E\left[ u\left( \Psi_T^{q_1q_2} \mid \Psi_T^{q_1q_2} = x \right) \right].
\]

For the Brownian motion risk model, the optimal reinsurance strategy for the aggregate claims process is

\[
q_{1t}^* = q_{2t}^* = \frac{2\theta}{2\theta + Ae^{(T-t)}}.
\]

5.3 Minimize Retained Risk with Respect to the Stop-loss Order

Denuit and Vermandele (1998) came up with three propositions regarding optimal reinsurance coverages for the ceding company when their optimality criterion is to minimize the retained risk with respect to the stop-loss order. The authors consider the stop-loss order because the treaty generating the smallest risk in the stop-loss sense will also be the most desirable with respect to fulfilling the common optimal reinsurance criteria (maximizing utility, minimizing variance, minimizing ruin probability, etc).

The authors assume the reinsurance premium principle used is the expected value principle and the reinsurance premium is fixed to \( \pi \). The reinsurance benefit must be a continuous, non-negative, non-decreasing function \( I(x) \), where \( 0 \leq I(x) \leq x, \ \forall X \in \mathbb{R}^+ \). Note that \( I(x) \) never increases faster than the claim amount itself and this set is denoted by \( \mathbb{C} \). Also, they introduced one proposition before discussing reinsurance strategies:

Proposition 1.1 (Denuit and Vermandele, 1998)

Let \( X_1, X_2, \ldots, X_n \) be exchangeable random variables and \( \psi_1, \psi_2, \ldots, \psi_n \) be measurable real functions. Define the function

\[
\bar{\psi}(x) = \frac{1}{n} \sum_{i=1}^{n} \psi_i(x), \ \forall x \in \mathbb{R}.
\]

Then,
\[ \sum_{i=1}^{n} \psi(X_i) \preceq_{cx} \sum_{i=1}^{n} \psi_i(X_i). \]

According to the above proposition, we can proceed to the three propositions regarding optimal reinsurance strategies.

**Proposition 1** Assume that the total claims \( S^{\text{ind}} \) of a given portfolio in a given period is given by

\[ S^{\text{ind}} = X_1 + X_2 + \cdots + X_n, \]

where \( n \) is the number of policies and the random variables \( X_i \) are non-negative and independent and identically distributed (i.i.d.). A global reinsurance treaty, \( I(X_1, X_2, \ldots, X_n) \) is measured in terms of \( S^{\text{ind}} \). The stop-loss reinsurance strategy is optimal.

**Proposition 2 (Global reinsurance is unavailable)** Assume that global reinsurance is not available and the markets only sell reinsurance in terms of \( I(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} I_i(x_i), \quad I_i \in \mathbb{C}. \)

The optimal reinsurance arrangement is such that \( I_i(x_i) = x_i \) and the excess-of-loss treaty is optimal.

**Proposition 3 (Collective Risk Model)** Assume \( N \) is the number of random claims occurring in a risk portfolio in a given period of time, and \( Z_i \) is the accidental amount of the \( i \)th claim, where \( i \leq N \). \( Z_i \) is non-negative and i.i.d. and \( N \) is non-negative. The aggregate claims amount of the collective risk model is

\[ S^{\text{coll}} = \sum_{i=1}^{N} Z_i, \quad I \in \mathbb{C} \text{ with } S^{\text{coll}} = 0 \text{ if } N = 0. \]

For the risk portfolio above, the stop-loss treaty is optimal. If claims are expressed as ‘individualized’ claims

\[ I(Z_1, Z_2, \ldots, Z_n) = \sum_{i=1}^{n} I_i(Z_i), \quad I_i \in \mathbb{C} \text{ with } I = 0 \text{ if } N = 0, \]

the excess-of-loss treaty with fixed deductible \( d \) is optimal.

**5.4 Maximize the Joint Survival Probability**

Considering the interests of both the insurer and reinsurer, Zhongfeng Qu (2012) developed the optimal retentions to maximize the joint survival probability under the combination of quota-
share and stop-loss reinsurance. The quota-share retention level is $b$ and the stop-loss retention limit is $d$. Ceded loss function is

$$I(x) = x - (bx \wedge d).$$

Assume that the quota-share level $b$ satisfies $0 < b < 1$, and the retention limit $d$ satisfies $d > 0$. The reinsurance premium principle used in this paper is expected value principle. The premium for reinsurer is

$$\Pi(X) = (1 + \theta)E[X - (bX \wedge d)].$$

The premium for insurer is $C(X) = (1 + \eta)E[X] - \Pi(X)$. $F_X(x)$ is the continuous distribution function and $S_X(x)$ is the survival function of $X$. Let $D$ denote the set of admissible retentions, then

$$D = \left\{(b, d): b \int_0^d S_X(x) \, dx > \frac{(\theta - \eta)\mu}{1 + \theta}\right\},$$

where $\mu = E[X]$. By analysis, we know that

$$b > \frac{(\theta - \eta)}{(1 + \theta)}, \quad d > d_0 \text{ for all } (b, d) \in D,$$

$$\inf \{b: (b, d) \in D\} = \frac{\theta - \eta}{1 + \theta}$$

and

$$\inf \{b: (b, d) \in D\} = d_0,$$

where $d_0$ is defined by

$$\int_{d_0}^{\infty} S_X(x) \, dx = \frac{(\theta + \eta)\mu}{1 + \theta}.$$

Assume $\eta < \theta$, $C(X) > 0$, the objective is to maximize

$$\Pr(b, d) = \Pr \{R_I(X) \leq u_I + C(X), I(X) \leq u_R + \Pi(X)\} = \Pr \left\{X \leq \frac{u_I + C(X)}{b}, X \leq \frac{u_R + \Pi(X)}{1 - b}, X \leq d\right\} + \Pr \{d \leq u_I + C(X), X \leq u_R + d + \Pi(X), X > \frac{d}{b}\},$$

where $u_I > 0$ and $u_R > 0$ are the initial wealth of the insurer and reinsurer respectively.

Let $G_1(b, d) = d + \Pi(X)$. If $S(d_0) \leq 1/(1 + \theta_R)$, then
\[
\inf_D G_1(b, d) = d_0 + (1 + \theta) \int_{d_0}^{\infty} S_X(x) \, dx.
\]

If \( S(d_0) > 1/(1 + \theta_R) \), then
\[
\inf_D G_1(b, d) = S^{-1} \left( \frac{1}{1 + \theta} \right) + (1 + \theta) \int_{S^{-1}(1/(1 + \theta))}^{\infty} S_X(x) \, dx.
\]

This paper summarizes the optimal retention levels under four different conditions:

1) If \( u_i \leq (\theta - \eta) \mu + (\theta - \eta) u_R/(1 + \theta) \) and \( \inf_D G_1(b, d) \geq u_i + (1 + \eta) \mu \), then there is no optimal level \((b^*, d^*)\) in \( D \).

2) If \( u_i \leq (\theta - \eta) \mu + (\theta - \eta) u_R/(1 + \theta) \) and \( \inf_D G_1(b, d) < u_i + (1 + \eta) \mu \), then the set of optimal levels \((b^*, d^*)\) is \( D^{(2)}_2 = \{(b, d) \in D : \frac{u_t + C(X)}{b} < d \} \), and the maximum value of the joint survival probability is
\[
\max_D P_{r_{joint}}(b, d) = P_{r_{joint}}(b^*, d^*) = F(u_i + u_R + (1 + \eta) \mu).
\]

3) If \( u_i > (\theta - \eta) \mu + (\theta - \eta) u_R/(1 + \theta) \) and \( \inf_D G_1(b, d) \geq u_i + (1 + \eta) \mu \), then the set of optimal levels \((b^*, d^*)\) is \( D^{(1)}_2 = \{(b, d) \in D : \frac{u_t + C(X)}{b} = \frac{u_R + \Pi(X)}{1 - b} \} \), and the maximum value of the joint survival probability is
\[
\max_D P_{r_{joint}}(b, d) = P_{r_{joint}}(b^*, d^*) = F(u_i + u_R + (1 + \eta) \mu).
\]

4) If \( u_i > (\theta - \eta) \mu + (\theta - \eta) u_R/(1 + \theta) \) and \( \inf_D G_1(b, d) < u_i + (1 + \eta) \mu \), then the set of optimal levels \((b^*, d^*)\) is \( D^{(3)}_{11} \cup D^{(3)}_{33} \), where
\[
D^{(3)}_{11} = \left\{(b, d) \in D : \frac{u_t + C(X)}{b} = \frac{u_R + \Pi(X)}{1 - b} \text{ and } \frac{u_t + C(X)}{b} < \frac{d}{b} \right\},
\]
\[
D^{(3)}_{33} = \left\{(b, d) \in D : \frac{d}{b} = \frac{u_t + C(X)}{b} \text{ and } \frac{d}{b} < \frac{u_R + \Pi(X)}{1 - b} \right\}
\]

and the maximum value of the joint survival probability is
\[
\max_D P_{r_{joint}}(b, d) = P_{r_{joint}}(b^*, d^*) = F(u_i + u_R + (1 + \eta) \mu).
\]
5.5 Minimize the Expected Time

Luo et al. (2016) considered a problem of minimizing the expected time to reach a goal for an insurance company whose surplus is relatively large compared to the size of an individual claim. They try to find an optimal reinsurance decision to minimize the expected time to achieve a goal, before reaching a lower barrier or a ruin level at which the insurance company has to be bankrupt.

Suppose two given surplus levels, $a$ is the ruin level where the insurance company gets bankrupt, and $b$ is the goal to reach. The reinsurance policy $\pi$ is called an admissible strategy. Define the stopping time of $X^\pi_t$ associated with an admissible strategy $\pi$, reaching $y$ where $b \geq y \geq a$ before reaching $a$ by

$$\tau^{a,y,\pi} = \inf\{t: X^\pi_t = y; b > X^\pi_s > a, \forall \ 0 < s < t\}.$$ 

Then they define the expected time of reaching the goal by

$$J^\pi(x) = E[\tau^{a,b,\pi} | X_0^\pi = x],$$

Denote the set of all admissible strategies by $\Pi$, the minimal expected time (or value function) determined by

$$f(x) = \inf_{\pi \in \Pi} J^\pi(x).$$

And the value function satisfies the following equation

$$\min_{\pi \in [0,1]} \left\{ [\mu - I(X)\alpha]V'(x) + \frac{[X - I(X)]^2\sigma^2}{2}V''(x) + 1 \right\} = 0,$$

where $\mu, \sigma$ are positive constants, $\alpha$ is quota share rate and $V(x) = f(x)$. They reach several conclusions considering two cases:

1) For non-cheap case ($\alpha > \mu$) and $a \geq -\infty$, they minimize the ruin probability:

   i) if $\mu < \alpha < 2\mu$, $V(x) = f(x) = \frac{e^{-c(x-a)} - e^{-c(b-a)}}{1-e^{-c(b-a)}}, Z(x) = \frac{2(\alpha - \mu)}{a},$ where $c = \frac{\alpha^2}{2\sigma^2(\alpha - \mu)}$.

   ii) if $\mu \geq 2\mu$, $V(x) = f(x) = \frac{e^{-d(x-a)} - e^{-d(b-a)}}{1-e^{-d(b-a)}}, Z(x) = 1,$ where $d = \frac{2\mu}{\sigma^2}$.

2) For cheap case ($\alpha = \mu$) and $a \geq -\infty$: 61
i) if $\frac{\sigma^2}{\mu} + a < b$, $Z(x) = \begin{cases} \frac{\mu}{\sigma^2} (x - a), & x \in \left(a, \frac{\sigma^2}{\mu} + a\right]; \\ 1, & x \in \left(\frac{\sigma^2}{\mu} + a, b\right]; \end{cases}$

ii) if $\frac{\sigma^2}{\mu} + a \geq b$, $Z(x) = \frac{\mu}{\sigma^2} (x - a)$.

3) For $\alpha \geq \mu$ and $a = -\infty$, without bankruptcy concern, the company would take the riskiest strategy with no reinsurance purchase: $V(x) = f(x) = \frac{b-x}{\mu}$ and $Z(x) = 1$. 
References


