Dividend Strategies for Insurance risk models

Project Report

1 Introduction

Based on different objectives, various insurance risk models with adaptive polices have been proposed, such as dividend model, tax model, model with credibility premium, and so on. In this report, we will only focus the study on dividend strategies.

Here are some reasons why the dividend strategy is of interest. For an insurance company, ruin occurs when a claim size is greater than its reserve, which means the insurer does not have enough assets to pay its liability. The goal for insurance company before was thus to minimize the probability of ruin. However, it is clear that, when facing the same claim risk, a senior company that starts business earlier has a smaller probability of ruin than a new company, only because more premiums have been collected. In other words, it is not necessary to require an insurance company to hold the reserves as many as possible. Also, when start a new business line, the insurance company may raise fund from its shareholders as the initial capital. Therefore, it is reasonable to pay the dividends as a return of their investment. Starting from De Finetti, researchers have shown that, in the classical risk model, the optimal strategy is the constant dividend barrier strategy in maximizing the total expected discounted dividends paid before ruin, if the distribution of the independent and identically distributed claims has a completely monotone density (see Gerber and Shiu (2006) and Loeffen (2008)). However, researchers also show that the constant dividend barrier strategy is not optimal, even by straying not too far from the above models; see, e.g., Albercher and Hartinger (2006). To be more specific, Albrecher and Hartinger (2006) discussed the non-optimality of the constant dividend barrier for the renewal risk model (Erlang-2 interclaim time) with exponential claims.

Insurers may pay out the excessive part to shareholders as dividends when the reserves are greater than certain amount, which is called barrier. There are different dividend strategies of interest in the literature (see Avanzi 2008). In the constant dividend strategy, dividends are paid continuously at the same rate as the premium whenever the surplus reaches (above) a pre-fixed barrier level. The threshold strategy is similar to the constant dividend strategy, but paying the dividends at a rate smaller than the the premium rate. Other strategies include linear (or nonlinear) barrier strategy, where the barrier level adjusted with respect to time, which is a nondecreasing linear (or nonlinear) function of time.
In this report, we intend to verify the theoretical findings using numerical examples and try to propose some suggestions for the non-optimal cases. There are several models considered in the following numerical example.

- M1: Compound Poisson risk model with exponential claims; under constant barrier
- M1*: Compound Poisson risk model with exponential claims; under linear dividend barrier
- M1**: Compound Poisson risk model with exponential claims; under nonlinear barrier
- M2: Compound Poisson risk model with gamma-distributed claims; under constant barrier
- M3: Renewal risk model (Gamma interclaim time) with exponential claims; under constant barrier

For each model, we will mainly simulate the expected present value of total dividend until ruin (we write it as EPV for short in the following), since most of the models have a ruin probability 1. The M1 (model 1) is used as a basis for comparison. M1* and M1** have linear and nonlinear barriers, which should have a lower EPV than M1. M2 has the claims whose density is not completely monotone, and M3 is a renewal process, where we know that the constant dividend barrier is not optimal. However, we do not know how the optimal strategy is like for M2 and M3, therefore, we will compare with the linear and nonlinear barriers as well.

Monte-Carlo simulation is the main approach to compare different models. Section 2 is the model description and some preliminary results. Section 3 is the numerical examples and comparison, with a conclusion in Section 4.

2 Model setup

Consider the renewal insurance risk process \( X = \{X_t\}_{t \geq 0} \) given by

\[
X_t = u + ct - S_t,
\]

where \( u \in \mathbb{R}, c > 0, \) and \( \{S_t\}_{t \geq 0} \) is a compound renewal process which is defined as

\[
S_t = \begin{cases} 
\sum_{i=1}^{N_t} P_{i}, & N_t > 0, \\
0, & N_t = 0,
\end{cases}
\]

where \( \{N_t\}_{t \geq 0} \) is a renewal process defined through the sequence of independent and identically distributed (iid) positive interarrival times \( \{T_i\}_{i \in \mathbb{N}} \) with distribution function
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\( K \) and \( \tilde{k} \), and \( \{P_i\}_{i \in \mathbb{N}} \) is a sequence of iid positive random variables (rv’s) with df \( P \) and \( \tilde{P} \), independent of \( \{N_t\}_{t \geq 0} \).

Furthermore, with the assumption of \( N(t) \) becomes a Poisson process, the model reduces to the traditional Compound Poisson risk model (see Gerber and Shiu, 1998). In the five comparative models mentioned above, we have two choices for the interclaim times and two choices for the claim severity, which are, respectively,

\[
\begin{align*}
\text{Exponential interclaim time:} & \quad k(x) = \lambda e^{-\lambda x}, \\
\text{Gamma interclaim time:} & \quad k(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x},
\end{align*}
\]

and

\[
\begin{align*}
\text{Exponential claim sizes:} & \quad p(x) = \mu e^{-\mu x}, \\
\text{Gamma claim sizes:} & \quad p(x) = \frac{\theta^v}{\Gamma(v)} x^{v-1} e^{-\theta x}.
\end{align*}
\]

Now we introduce the calculation for EPV. Let \( \{D_t\}_{t \geq 0} \) be the total dividends paid to the shareholders by time \( t \), and \( \{U_t\}_{t \geq 0} \) be the company’s surplus after dividends payment, so

\[ U_t = X_t - D_t. \]

Define the time to ruin

\[ T = \inf \{ t \geq 0 : U_t < 0 \}, \]

and the (infinite-time) ruin probability

\[ \psi(u) = P(T < \infty). \]

Define the EPV

\[ D = E[\int_0^T e^{-\delta t} dD_t], \]

where \( \delta \) is the discounted factor (or force of interest).

The barrier strategies are defined as follows

- Constant dividend barrier

\[
dD_t = \begin{cases} 
U_t - b, & U_t > b, \\
cdt, & U_t = b, \\
0, & U_t < b.
\end{cases}
\]

- Linear Barrier \( b_t = b + at \)

\[
dD_t = \begin{cases} 
U_t - (b + at), & U_t > b + at, \\
(c - a)dt, & U_t = b + at, \\
0, & U_t < b + at.
\end{cases}
\]
• Nonlinear Barrier $b_t = f(t)$

$$dD_t = \begin{cases} U_t - f(t), & U_t > f(t), \\ c dt - df(t), & U_t = f(t), \\ 0, & U_t < f(t), \end{cases}$$

where $f(t)$ could be any function, including constant, linear and nonlinear. For example, we could choose the nonlinear barrier to be $b_t = be^{it}$ which can be considered as a result of the inflation effect. Other nonlinear barrier of interest can be found in Albrecher et al. (2003). We may also consider the ratio of $a/b$ in the liner barrier model as the simple interest rate.

Note that the above two figures (Fig 1 and Fig 2) are from Avanzi (2008). They are used to illustrate one sample path for $U_t$ and $D_t$ under the constant dividend barrier (Fig 1) and linear barrier (Fig 2).

We now present some analytical results. These mathematical results will not only provide the insight on risk management, but also set a benchmark in comparison within different strategies.

• If there is no barrier, the ruin probability for compound Poisson risk model with exponential claim (see Gerber and Shiu, 1998):

$$\psi(u) = \frac{\lambda}{c \mu} e^{(\lambda/c - \mu)u}. \quad (2.3)$$

• Under the constant dividend barrier, it has been shown that the (infinite) ruin probability is 1, as the ruin time is a finite valued random variable (see Lin et al., 2003). The EPV can be calculated under the compound Poisson risk model with
exponential claims (actually for a more general spectrally negative Lévy process, see Loeffen (2008), Kyprianou (2013) and reference therein):

\[ D = \frac{W^{(\delta)}(u)}{W^{(\delta)}(b)}, \quad u \leq b. \]

and the optimal level \( b^\ast \) can be found:

\[ b^\ast = \sup \{ b \geq 0 : W^{(\delta)}(b) \leq W^{(\delta)}(x) \text{ for all } x \in [u, \infty) \}. \]

The scale function of the compound Poisson risk model with exponential claims is

\[ W^{(\delta)}(x) = \frac{1}{c} \left\{ \frac{\rho + \mu}{\rho + R} e^{\rho x} + \frac{R - \mu}{\rho + R} e^{-Rx} \right\}. \]

where \( \rho > 0 \) and \( -R < 0 \) are the two roots to the equation

\[ cz - \frac{\lambda z}{z + \mu} = \delta. \]

Therefore, after some calculation, we find

\[ D = \frac{(\rho + \mu)e^{\rho u} + (R - \mu)e^{-Ru}}{\rho(\rho + \mu)e^{\rho b} - R(R - \mu)e^{-Rb}}, \quad (2.4) \]

and

\[ b^\ast = \frac{1}{\rho + R} \ln \frac{R^2(\mu - R)}{\rho^2(\mu + \rho)}. \quad (2.5) \]

### 3 Numerical examples

For insurance companies, the parameters in the models can be set using statistical estimations. We choose the reasonable parameters in this section. We set the time boundary as 1000, which means we stop the simulation at time 1000 if ruin does not occur before that.

Under each barrier strategy, we set up an initial reserve \( u \), an initial barrier \( b \), a constant premium rate \( c \), mean for interclaim time and claim size. We use Matlab for the following simulations.

**Example 3.1** Choose \( u = 0 \), \( \lambda = 1 \), \( \mu = 1 \), \( \delta = 0.03 \), \( c = 1.1 > 1 \). We can calculate the optimal barrier \( b^\ast = 1.2338 \) and the maximized EPV is \( D^\ast = 1.0908 \). And the infinite-time ruin probability with no barrier is \( \psi(u) = 0.9091 \).
### 3.1 Constant dividend barrier

With the the parameters in Example 3.1, the simulation results are summarized below. Table 1 shows the optimality of the barrier $b^*$. Note by $\text{Exp}(\mu)$ we mean an exponential distribution with mean $1/\mu$.

In Table 2, we change the exponential distribution assumptions to Gamma distribution as in M2 and M3, and the parameters are picked to match the mean. Table 2 shows that when assumptions changed a little bit, even with same expectation, the original $b^*$ is not an optimal barrier (among all the constant barrier) anymore.

Next subsection, we will consider the linear and nonlinear barriers, to see whether there is any other optimal choice. Also, the simulated ruin probability is always 1.

### 3.2 Linear and nonlinear barriers

As for this two barriers, we need two new parameters to describe the barriers, the slope $a$ and the increasing force (or force of interest) $i$.

Continue from Example 3.1. We will find the (finite-time) ruin probability, and the corresponding expected present value of dividends. We put a more detailed simulation process in Appendix for illustration.

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**Table 1:** Compound Poisson with exponential claims under Constant barrier

<table>
<thead>
<tr>
<th>Interclaim time</th>
<th>Claim size</th>
<th>barrier</th>
<th>EPV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exp(1)</td>
<td>Exp(1)</td>
<td>1.2338</td>
<td>1.0937</td>
</tr>
<tr>
<td>Exp(1)</td>
<td>Exp(1)</td>
<td>2</td>
<td>1.0883</td>
</tr>
<tr>
<td>Exp(1)</td>
<td>Exp(1)</td>
<td>1.5</td>
<td>1.0868</td>
</tr>
<tr>
<td>Exp(1)</td>
<td>Exp(1)</td>
<td>1</td>
<td>1.0903</td>
</tr>
</tbody>
</table>

**Table 2:** Constant barrier for M2 and M3

<table>
<thead>
<tr>
<th>Interclaim time</th>
<th>Claim size</th>
<th>barrier</th>
<th>EPV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exp(1)</td>
<td>Gamma(2,2)</td>
<td>1.2338</td>
<td>0.8457</td>
</tr>
<tr>
<td>Exp(1)</td>
<td>Gamma(2,2)</td>
<td>2</td>
<td>0.8380</td>
</tr>
<tr>
<td>Exp(1)</td>
<td>Gamma(2,2)</td>
<td>1.5</td>
<td>0.8386</td>
</tr>
<tr>
<td>Exp(1)</td>
<td>Gamma(2,2)</td>
<td>1</td>
<td>0.8477</td>
</tr>
<tr>
<td>Exp(1)</td>
<td>Gamma(2,2)</td>
<td>0.5</td>
<td>0.8740</td>
</tr>
<tr>
<td>Gamma(2,2)</td>
<td>Exp(1)</td>
<td>1.2338</td>
<td>1.1242</td>
</tr>
<tr>
<td>Gamma(2,2)</td>
<td>Exp(1)</td>
<td>2</td>
<td>1.1232</td>
</tr>
<tr>
<td>Gamma(2,2)</td>
<td>Exp(1)</td>
<td>1.5</td>
<td>1.1258</td>
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<tr>
<td>Gamma(2,2)</td>
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<td>1</td>
<td>1.1156</td>
</tr>
<tr>
<td>Gamma(2,2)</td>
<td>Exp(1)</td>
<td>0.5</td>
<td>1.0969</td>
</tr>
</tbody>
</table>
From Table 3, we can see that:

- M1* would not yield a larger EPV than M1 (with optimal $b^*$), which verifies the optimality of the constant barrier for the classical model.

- In this example, we could find a linear barrier strategy for M2 with a higher EPV than the constant barriers considered in the previous subsection, but we are not sure whether this holds in general. This is not true for M3.

- For all the models, there is a tradeoff between the EPV and ruin probability.

Now we move to the nonlinear barrier case. We first choose the nonlinear barrier as $b_t = be^{it}$, which is called as the exponential barrier. Analyzing the result, we find that for linear barrier and exponential barrier, the increase rate, like $a/b$ or $i$, is critical to determine the EPV of dividends. If the increase rate is set too large, there is little chance for the reserve $u$ to exceed the barrier, which should result in a small EPV of dividends. Especially for exponential barrier, since the barrier increases at a much higher speed, even without a large claim happening, hardly can the reserve to “catch up” with the barrier again. In order to avoid this, we think an increasing concave barrier makes more sense to discuss here.

Note that for the nonlinear barrier (such as the exponential barrier, we also need to consider whether the barrier and reserve will cross each other twice in one interclaim time. We illustrate the situations in Fig 3 and Fig 4.
Continue from Example 3.1. We further assume the nonlinear barrier $b_t = b e^{it}$. It is not hard to check that this function is an increasing function for all $t$, and a concave function for $t < 1/i^2$. Since $i$ is a small number less than 1, this assumption valid for the increasing concave assumption before the fix terminal time 1000. Actually, we only need to assume $i < (1/1000)^{1/2} = 0.0316$. Another good fact about this barrier is that it will avoid the double crossing case shown in Fig 4, if we choose the proper parameters for $b$ and $i$ (to make the slope less then the premium rate $c$). For example, when we choose $b = 2$ and $i = 0.02$, see Fig 5.

Fig 3: the EPV integration is from solution $x$ to $t_0+dt$

Fig 4: There are two solutions between $t_0$ and $t_0+dt$, the shaded area is paid dividend

Fig 5: A nonlinear barrier example
Table 4: Non-linear barrier: $b_t = b e^{i \sqrt{t}}$ with $i = 0.02$

<table>
<thead>
<tr>
<th>Interclaim time</th>
<th>Claim size</th>
<th>b</th>
<th>EPV</th>
<th>Ruin</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exp(1)</td>
<td>Exp(1)</td>
<td>2</td>
<td>1.0197</td>
<td>1</td>
</tr>
<tr>
<td>Exp(1)</td>
<td>Exp(1)</td>
<td>1.5</td>
<td>1.0688</td>
<td>1</td>
</tr>
<tr>
<td>Exp(1)</td>
<td>Exp(1)</td>
<td>1</td>
<td>1.0907</td>
<td>1</td>
</tr>
<tr>
<td>Exp(1)</td>
<td>Exp(1)</td>
<td>0.5</td>
<td>1.0608</td>
<td>1</td>
</tr>
<tr>
<td>Exp(1)</td>
<td>Gamma(2,2)</td>
<td>2</td>
<td>0.8705</td>
<td>1</td>
</tr>
<tr>
<td>Exp(1)</td>
<td>Gamma(2,2)</td>
<td>1.5</td>
<td>0.8786</td>
<td>1</td>
</tr>
<tr>
<td>Exp(1)</td>
<td>Gamma(2,2)</td>
<td>1</td>
<td>0.8440</td>
<td>1</td>
</tr>
<tr>
<td>Exp(1)</td>
<td>Gamma(2,2)</td>
<td>0.5</td>
<td>0.8710</td>
<td>1</td>
</tr>
<tr>
<td>Exp(1)</td>
<td>Gamma(2,2)</td>
<td>0.3</td>
<td>0.9094</td>
<td>1</td>
</tr>
<tr>
<td>Gamma(2,2)</td>
<td>Exp(1)</td>
<td>2</td>
<td>1.1421</td>
<td>1</td>
</tr>
<tr>
<td>Gamma(2,2)</td>
<td>Exp(1)</td>
<td>1.5</td>
<td>1.1132</td>
<td>1</td>
</tr>
<tr>
<td>Gamma(2,2)</td>
<td>Exp(1)</td>
<td>1</td>
<td>1.1476</td>
<td>1</td>
</tr>
<tr>
<td>Gamma(2,2)</td>
<td>Exp(1)</td>
<td>0.5</td>
<td>1.1125</td>
<td>1</td>
</tr>
</tbody>
</table>

From Table 4, we see that:

- M1** would not yield a larger EPV than M1 (with optimal $b^*$), which verifies the the optimality of the constant barrier for the classical model.

- This concave barrier make the ruin probability 1. It could be explained by the low increasing speed of the barrier, as illustrated in Fig 5.

- The EPVs for M2 and M3 are larger with the nonlinear barrier than those in the linear barrier that we found. There might be explained: in this example, the nonlinear barrier proposed is lower than the linear barrier, so we expect more dividend paid at the beginning, with a low discount factor, i.e., a high present value.

We also have done some parameters’ sensitivity analysis, and the findings are straightforward and consistent with the intuition. For example, under the compound Poisson risk model with exponential claim and constant barrier, the EPV is an increasing function of the premium rate $c$ and a decreasing function of the mean of the claim severity. Another comparison done is about the constant barrier with different Gamma interclaim time; see Table 5. We want to see the effect of the variance($= \alpha^2/\beta$) of interclaim time to the EPV in the renewal model M3. Compared with Table 2, we find that the higher the variance is, the lower the EPV is.
Table 5: Different Gamma interclaim times

<table>
<thead>
<tr>
<th>Interclaim time</th>
<th>Claim size</th>
<th>b</th>
<th>EPV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gamma(0.5,0.5)</td>
<td>Exp(1)</td>
<td>2</td>
<td>1.0368</td>
</tr>
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<td>Gamma(0.5,0.5)</td>
<td>Exp(1)</td>
<td>1.5</td>
<td>1.0407</td>
</tr>
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<td>Gamma(0.5,0.5)</td>
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<td>1</td>
<td>1.0529</td>
</tr>
<tr>
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<td>Exp(1)</td>
<td>0.5</td>
<td>1.0635</td>
</tr>
<tr>
<td>Gamma(8,8)</td>
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<td>2</td>
<td>1.1834</td>
</tr>
<tr>
<td>Gamma(8,8)</td>
<td>Exp(1)</td>
<td>1.5</td>
<td>1.1648</td>
</tr>
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<td>Gamma(8,8)</td>
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<td>1</td>
<td>1.1463</td>
</tr>
<tr>
<td>Gamma(8,8)</td>
<td>Exp(1)</td>
<td>0.5</td>
<td>1.1079</td>
</tr>
</tbody>
</table>

4 Conclusion

In this report, we focus on the dividend strategies for classical risk model and renewal risk model. We learn that for the compound Poisson risk model with exponential claim sizes, the constant dividend barrier will maximize its EPV. It might be hard to answer whether there is an optimal dividend strategy for the classical risk model with the non-completely monotone distributed claims or the renewal risk models. To answer this question, the objective function that we choose does matter, which might lead to a Yes or No. Also, we do see the tradeoff between the EPV and ruin probability. Future research could be done on a topic where we restrict the ruin probability to be less than 1, and find the optimal strategy for the traditional risk model.

Appendix: Ruin probability simulation with nonlinear barrier

Variables created in this function include initial barrier $b$, initial reserve $u$, barrier’s force of interest $i$, Poisson process parameter $\lambda$, claim size mean $1/\mu$ and premium rate $c$. We first simulate the time interval of claim occurrence using $dt = -\ln(rand(1,1))/\lambda$, as it follows exponential distribution. We initialize $t$ to be 0, so the time of occurrence can be expressed as $t = t + dt$. We then simulate the process using a while loop when $t$ is less than 1000. In every occurrence of claim, we first simulate a claim size by generating a random exponential number with mean $1/\mu$ using function exprnd(). Then we compare $U_t$ and $b_t$ to decide if barrier has been reached. If it does, the reserve is equal to $b_t$, and is equal to $U_t$ if it doesn’t. Then we compare the reserves with the claim size. If the claim is greater than the reserves at time $t$, the while loop terminates, and the indicator variable $p$ is assigned with value 1, indicating a ruin occurred; otherwise, if no ruin happens during the while-loop, 0 is assigned to $p$. 


indicating no ruin occurs. Function Loopsum runs the procedure above for \( n = 10000 \) times and sum all \( p \) values. Dividing the sum of \( p \) values by \( n \), we then derive the probability of ruin. There is a flow chart demonstrating the process below.

![Flow chart illustration for ruin probability simulation](image)

**References**


