Practical Implementation of Optimal Reinsurance under Distortion Risk Measures

Shuyu Guo, Michael Kim, Bowen Shan, and Utsav Thota

Institution: University of Illinois at Urbana-Champaign, Department of Mathematics

May 24, 2018

1 Introduction

As a way to alleviate the financial burden of a loss on an individual, an insurance company may assume the loss. To prevent insolvency, the insurance company may choose to cede part of its risk to a reinsurance company, which functions in a similar way as the insurer. An insurance company will look to optimize the amount of loss ceded to a reinsurance company and lower their premium payment by identifying which reinsurance contract to purchase in each situation.

Optimal reinsurance research has been of recent interest and there has been numerous studies conducted, each with their own methodology. Our goal for this research is to investigate different distortion risk measures as various criteria for optimal reinsurance contracts, which will minimize the overall loss to the insurer. In this paper, we specifically look at the distortion risk measures of value-at-risk (VaR) and conditional-tail expectation (CTE), before moving on to a more general approach.

We first summarize the findings in the paper, "Optimal Reinsurance under VaR and CTE Risk Measures" by Cai et al. (2008), referencing the concepts covered in Chapters 1 and 2 of "Actuarial Theory for Dependent Risks" by Denuit et al. (2005). Next, we summarize the findings in "Optimal Reinsurance Revisited – A Geometric Approach" by Cheung (2010) and "Characterizations of Optimal Reinsurance Treaties: A Cost-Benefit Approach" by Cheung and Lo (2015), which make an attempt at aggregating and simplifying previously investigated ideas. Our motivation for this research is to practically implement the theory in the reinsurance market. For each paper we cover, we conduct a numerical analysis based on the theorems provided. This will give insurers insight into how these theorems can be practically implemented.

2 Preliminaries

2.1 Basic Concepts of Risk

A risk X is a non-negative random variable (RV). It represents the loss of the insurer or amount of money needed to indemnify policyholders.

The distribution function (df) of the RV X, denoted by F_X is defined as:

$$F_X(x) = Pr\left[X^{-1}\left((-\infty, x]\right)\right] \equiv Pr[X \leq x], \quad x \in \mathbb{R}.$$

This measures the probability that the RV X is less than or equal to x. Conversely, the survival function $S_X(x)$ is defined as follows:

$$S_X(x) = 1 - F_X(x) = Pr[X > x], \quad x \in \mathbb{R}.$$

Therefore, $S_X(x)$ represents the probability that X assumes a value larger than x. For example, if X is the random future lifetime of a policyholder, then $S_X(x)$ is the probability that the policyholder survives up to age x. Denote S_X^{-1} as the left inverse function of S_X.

In addition, given a df F_X, we define the inverse functions F_X^{-1} and F_X^{-1+} of F_X. For any real number x and probability level p:

$$F_X^{-1} = \inf\{x \in \mathbb{R} | F_X(x) \geq p\} = \sup\{x \in \mathbb{R} | F_X(x) < p\},$$

$$F_X^{-1+} = \inf\{x \in \mathbb{R} | F_X(x) > p\} = \sup\{x \in \mathbb{R} | F_X(x) \leq p\},$$

where inf denotes the infimum, which represents the greatest lower bound, and sup denotes the supremum, which represents the least upper bound.

For any real number x and probability level p, the following equivalences hold:

(i) $F_X^{-1}(p) \leq x \Leftrightarrow p \leq F_X(x)$,

(ii) $x \leq F_X^{-1+}(p) \Leftrightarrow Pr[X < x] = F_X(x^-) \leq p$.

2.2 Risk Measures

Measuring risk/loss is equivalent to establishing a correspondence Q between the space of RVs and non-negative real number set $\mathbb{R}_+$. The real number denoting a general risk measure associated with X will henceforth be denoted as $Q(X)$. Thus, a risk measure is nothing but a function that assigns a nonnegative real number to a risk.

There are several properties that may be fulfilled by a risk measure:

1. Non-excessive loading (no-rip off)
$$Q(X) \leq \sup(X). \tag{2.2.1}$$

2. Non-negative loading
$$Q(X) \geq \mathbb{E}(X). \tag{2.2.2}$$

3. Translativity
$$Q(X+c) = Q(X) + c, \tag{2.2.3}$$
for each constant c.

4. Constancy (No unjustified loading)
$$Q(c) = c, \tag{2.2.4}$$
for each constant c.

5. Subadditivity
$$Q(X+Y) \leq Q(X) + Q(Y) \quad \forall\ X, Y. \tag{2.2.5}$$

6. Comonotonic additivity
$$Q(X+Y) = Q(X) + Q(Y), \tag{2.2.6}$$
when RVs X and Y are comonotonic. X and Y are comonotonic if $X = f(Z)$ and $Y = g(Z)$ with Z being a RV and f, g are non-decreasing.

7. Positive homogeneity
$$Q(cX) = cQ(X), \tag{2.2.7}$$
for any positive constant c.

8. Monotonicity
$$X \leq Y \Rightarrow Q(X) \leq Q(Y). \tag{2.2.8}$$

9. Continuity with respect to convergence in distribution

   Let $\{X_n, n = 1, 2, \dots\}$ be a sequence of risks such that $X_n \xrightarrow{d} X$ as $n \to +\infty$ such that $\lim_{n\to+\infty} F_{X_n}(x) = F_X(x)$ for every continuity point x of F_X
$$\Rightarrow \lim_{n\to+\infty} Q[X_n] = Q(X). \tag{2.2.9}$$

10. Convexity
$$Q(\lambda X + (1-\lambda)Y) \leq \lambda Q(X) + (1-\lambda)Q(Y), \tag{2.2.10}$$
for $0 \leq \lambda \leq 1$.

A distortion risk measure has a non-decreasing function $g : [0,1] \to [0,1]$ with $g(0) = 0$ and $g(1) = 1$. The distorted expectation is:

$$\rho_g(X) = -\int_{-\infty}^{0} (1 - g(S_X(x)))\ dx + \int_{0}^{\infty} g(S_X(x))\ dx.$$

The function g is called a distortion because it distorts the probabilities in terms of the survival function.

We consider two distortion risk measures in particular: value-at-risk (VaR) and conditional-tail-expectation (CTE).

1. The VaR of X at a confidence level $1-\alpha$, where $0<\alpha<1$ is

$$\mathrm{VaR}_X(\alpha) = \inf\{x : Pr\{X > x\} \leq \alpha\}.$$

   If $\alpha \geq S_X(0)$, then $1-\alpha \leq 1-S_X(0) = F_X(0)$, and $F_X(x) = Pr(X > x) \geq 1-\alpha$ holds for all $x>0$, and $\mathrm{VaR}_X(\alpha) = 0$. If $0<\alpha<S_X(0)$ and X has a continuous strictly increasing df on $(0,\infty)$, then $\mathrm{VaR}_X(\alpha) = \inf\{x : S_X(x) \leq \alpha\} = S_X^{-1}(\alpha)$. VaR satisfies translativity, positive homonogeneity, and is comonotonic additive (items 2.2.3, 2.2.7, 2.2.6).

2. The CTE of X at a confidence level $1-\alpha$, where $0<\alpha<1$ is

$$\mathrm{CTE}_X(\alpha) = \mathbb{E}[X : X \geq \mathrm{VaR}_X(\alpha)] = \mathrm{VaR}_X(\alpha) + \frac{1}{1-F_X(\mathrm{VaR}_X(\alpha))}\mathbb{E}[(X - \mathrm{VaR}_X(\alpha))_+].$$

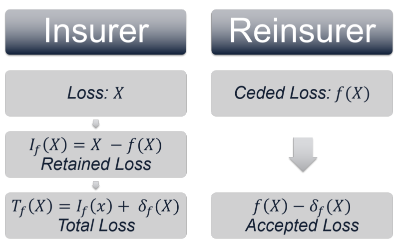
   CTE is equal to TVaR when F_X is continuous. CTE satisfies translativity, monotonicity, positive homogeneity, subadditivity, and is comonotonic additive (items 2.2.3, 2.2.5, 2.2.6, 2.2.7, 2.2.8).

The coherent risk measure defined by Artzner et al. (1999) must be translative, positive homogeneous, subadditive and monotone (items 2.2.3, 2.2.5, 2.2.7, 2.2.8). The convex risk measure defined by Föllmer and Schied (2002) must satisfy translativity, monotonicity and convexity (items 2.2.3, 2.2.8, 2.2.10). This makes CTE a coherent risk measure only when F_X if continuous, but not VaR. Neither VaR nor CTE is a convex risk measure.

2.3 Basic Concepts of Reinsurance

Reinsurance is an employed risk management strategy to protect the insurer against potentially large losses. Recall that X is the initial loss taken by an insurer, with df $F_X(x)$ and survival function $S_X(x)$. Under a reinsurance policy, the insurer cedes part of its loss, $f(X)$, where $0 \leq f(X) \leq X$, to a reinsurer. Thus the insurer has a retained loss $I_f(X) = X - f(X)$ where $f(x)$ is the ceded loss function and $I_f(x)$ is the retained loss function.

Keep in mind that the insurer needs to pay a reinsurance premium to the reinsurer which incurs an additional risk. Let $\delta_f(X)$ be the reinsurance premium, where under the expectation premium principle: $\delta_f(X) = (1+\rho)\mathbb{E}[f(X)]$, and $\rho > 0$ is a safety loading. Let $T_f(X)$ denote the total risk of the insurer, where $T_f(X) = I_f(X) + \delta_f(X)$.



2.4 Reinsurance Contracts

We discuss three commonly used reinsurance design models which define $f(X)$: quota-share reinsurance, stop-loss reinsurance, and change-loss reinsurance. The optimal solutions under VaR and CTE take these three forms.

1. In terms of quota-share reinsurance, we have $f(x) = ax$ and $I_f(x) = x - f(x) = x - ax = (1-a)x$, where $0 \leq a \leq 1$.

2. In terms of stop-loss reinsurance, we have $f(x) = (x-d)_+ = \max\{0, x-d\}$ and $I_f(x) = x - f(x) = x - \max\{0, x-d\} = \min\{x, d\}$, where $d \geq 0$.

3. In terms of change-loss reinsurance, we have $f(x) = a(x-d)_+$ and $I_f(x) = x - f(x) = (1-a)x + a\min\{x, d\}$, where $0 \leq a \leq 1$, where $d \geq 0$.

3 Optimal Reinsurance under VaR and CTE Risk Measures

3.1 VaR - Classical Approach

To begin, we define $\mathcal{F}$ to be the class of ceded loss functions consisting of all increasing convex functions $f(x)$ defined on $[0,\infty)$ and satisfying $0 \leq f(x) \leq x$ for $x \geq 0$ but excluding $f(x) \equiv 0$. Because all function in $\mathcal{F}$ are convex, all functions in $\mathcal{F}$ are continuous on $[0,\infty)$.

We can construct optimal reinsurance models depending on risk measure methods defined below, otherwise known as the objective functions:

VaR-optimization:

$$\mathrm{VaR}_{T_{f^*}(X)}(\alpha) = \min_{f \in \mathcal{F}}\{\mathrm{VaR}_{T_f(X)}(\alpha)\} \tag{3.1.1}$$

CTE-optimization:

$$\mathrm{CTE}_{T_{f^*}(X)}(\alpha) = \min_{f \in \mathcal{F}}\{\mathrm{CTE}_{T_f(X)}(\alpha)\} \tag{3.1.2}$$

We continue by considering a specific subclass of $\mathcal{F}$, class $\mathcal{H}$, which consists of all non-negative functions $h(x)$ defined on $[0,\infty)$, where:

$$h(x) = \sum_{j=1}^{n} c_{n,j}(x - d_{n,j})_+, \quad x \geq 0, \ n = 1, 2, \ldots$$

and where $c_{n,j} \geq 0, d_{n,j} \geq 0$ are constants such that:

$$0 < \sum_{j=1}^{n} c_{n,j} \leq 1 \quad \text{and} \quad 0 \leq d_{n,1} \leq d_{n,2} \leq \cdots \leq d_{n,n} \quad \text{for all} \quad n = 1, 2, \ldots$$

Here:

$$h(x) = c_{n,1}(x - d_{n,1})_+ + c_{n,2}(x - d_{n,2})_+ + \cdots + c_{n,n}(x - d_{n,n})_+$$

is the sum of a sequence of ceded loss functions under change-loss reinsurance.

Note that any function in $\mathcal{F}$ is the limit of a sequence of functions in $\mathcal{H}$. Thus, the optimal functions in $\mathcal{H}$ are also optimal functions in $\mathcal{F}$ which minimize the VaR and CTE of the total cost $T_f(X)$ for $f \in \mathcal{F}$. As a result, we determine the optimal ceded loss functions that minimize VaR and CTE of the total cost $T_h(X)$ in class $\mathcal{H}$ as defined above.

Since $T_f(X) = I_f(X) + \delta_f(X)$, and following from translativity (2.2.3) of VaR and CTE:

$$\mathrm{VaR}_{T_f(X)}(\alpha) = \mathrm{VaR}_{I_f(X)}(\alpha) + \delta_f(X) \tag{3.1.3}$$

$$\mathrm{CTE}_{T_f(X)}(\alpha) = \mathrm{CTE}_{I_f(X)}(\alpha) + \delta_f(X) \tag{3.1.4}$$

Under the assumption that the reinsurance premium is determined using the expectation premium principle, for any function $h(x) = \sum_{j=1}^{n} c_{n,j}(x - d_{n,j})_+ \in \mathcal{H}$, the reinsurance premium of the ceded loss $h(X)$ can be written as:

$$\delta_h(X) = (1+\rho)\mathbb{E}[h(X)] = (1+\rho)\left\{\sum_{j=1}^{n} c_{n,j} \int_{d_{n,j}}^{\infty} S_X(x)\,dx\right\}$$

where:

$$\int_{d_{n,j}}^{\infty} S_X(x)\,dx = \mathbb{E}[(x - d_{n,j})_+]$$

The retained loss function becomes:

$$I_h(X) = X - h(X) = X - \sum_{j=1}^{n} c_{n,j}(X - d_{n,j})_+$$

Therefore, $I_h(X)$ can be separated by three parts depending on the value of X. We define $A_{n,i} = 1 - \sum_{j=1}^{i} c_{n,j}$, and $B_{n,i} = \sum_{j=1}^{i} c_{n,j} d_{n,j}$, for $i = 1, 2, \ldots, n$.

1. If $X \leq d_{n,1}$, $c_{n,j}(X-d_{n,j})_+$ becomes 0, since $d_{n,j}$ increases as j increases. Hence, $I_h(X) = X$.
2. If $d_{n,i} \leq X \leq d_{n,i+1}$, where $i = 1, \ldots, n-1$, $c_{n,j}(X - d_{n,j})_+$ becomes 0 if X is greater or equal to $d_{n,i+1}$. Hence, $I_h(X) = X - \sum_{j=1}^{i} c_{n,j}(X - d_{n,j})_+ = A_{n,i}X + B_{n,i}$
3. If $X \geq d_{n,n}$, all values of $c_{n,j}(X - d_{n,j})_+$ are greater than 0. Hence, $I_h(X) = X - \sum_{j=1}^{n} c_{n,j}(X - d_{n,j})_+ = A_{n,n}X + B_{n,n}$

To summarize:

$$I_h(X) = \begin{cases} X, & X \leq d_{n,1} \\ A_{n,i}X + B_{n,i}, & d_{n,i} \leq X \leq d_{n,i+1}, \quad i = 1, \ldots, n-1 \\ A_{n,n}X + B_{n,n}, & d_{n,n} \leq X \end{cases}$$

To get the survival function, $S_{I_h(X)}(x)$, we can use $P(A_{n,i}X + B_{n,i} > x) = P\left(X > \frac{x-B_{n,i}}{A_{n,i}}\right)$ and the coefficients of $I_h(X)$ satisfy $A_{n,i}d_{n,i+1} + B_{n,i} = A_{n,i+1}d_{n,i+1} + B_{n,i+1}$. The survival function can be divided into three parts depending on the value of x.

1. If $x \leq d_{n,1}$, $S_{I_h(X)}(x) = 1 - F_{I_h(X)}(x) = S_X(x)$
2. If $A_{n,i}d_{n,i} + B_{n,i} \leq x \leq A_{n,i}d_{n,i+1} + B_{n,i}$,
   $S_{I_h(X)}(x) = 1 - F_{I_h(X)}(x) = P(A_{n,i}X + B_{n,i} > x) = P\left(X > \frac{x-B_{n,i}}{A_{n,i}}\right) = S_X\left(\frac{x-B_{n,i}}{A_{n,i}}\right)$
3. If $x \geq A_{n,n}d_{n,n} + B_{n,n}$,
   $S_{I_h(X)}(x) = 1 - F_{I_h(X)}(x) = P(A_{n,n}X + B_{n,n} > x) = P\left(X > \frac{x-B_{n,n}}{A_{n,n}}\right) = S_X\left(\frac{x-B_{n,n}}{A_{n,n}}\right)$

Therefore:

$$S_{I_h(X)}(x) = \begin{cases} S_X(x), & x \leq d_{n,1} \\ S_X\left(\frac{x-B_{n,i}}{A_{n,i}}\right) & A_{n,i}d_{n,i} + B_{n,i} \leq x \leq A_{n,i}d_{n,i+1} + B_{n,i} \\ S_X\left(\frac{x-B_{n,n}}{A_{n,n}}\right) & x \geq A_{n,n}d_{n,n} + B_{n,n}, \end{cases}$$

where $S_X\left(\frac{x-B_{n,i}}{A_{n,i}}\right) = S_X(\infty) = 0$ when $A_{n,i} = 0$ for some $i = 1, \ldots, n$.

Using the definition of S_X^{-1}:

$$S_{I_h(X)}^{-1}(\alpha) = \mathrm{VaR}_{I_h(X)}(\alpha) = \begin{cases} S_X^{-1}(\alpha), & S_X^{-1}(\alpha) \leq d_{n,1} \\ A_{n,i}S_X^{-1}(\alpha) + B_{n,i} & d_{n,i} \leq S_X^{-1}(\alpha) \leq d_{n,i+1}, i = 1, \ldots, n-1 \\ A_{n,n}S_X^{-1}(\alpha) + B_{n,n} & d_{n,n} \leq S_X^{-1}(\alpha) \end{cases}$$

From (3.1.3):

$$\mathrm{VaR}_{T_h(X)}(\alpha) = \begin{cases} S_X^{-1}(\alpha) + \delta_h(X), & S_X^{-1}(\alpha) \leq d_{n,1} \\ A_{n,i}S_X^{-1}(\alpha) + B_{n,i} + \delta_h(X) & d_{n,i} \leq S_X^{-1}(\alpha) \leq d_{n,i+1}, i = 1, \ldots, n-1 \\ A_{n,n}S_X^{-1}(\alpha) + B_{n,n} + \delta_h(X) & d_{n,n} \leq S_X^{-1}(\alpha) \end{cases}$$

We now introduce some notation that will be useful later in the paper.

$$\rho^* = \frac{1}{1+\rho}$$

$$d^* = S_X^{-1}(\rho^*)$$

$$g(x) = x + \frac{1}{\rho^*}\int_x^\infty S_X(t)\,dt, \;\; x \geq 0$$

$$u(x) = S_X^{-1}(x) + \frac{1}{\rho^*}\int_{S_X^{-1}(x)}^\infty S_X(t)\,dt, \;\; x \geq 0$$

We begin with the following lemma:

Lemma 3.1. *For any $f \in \mathcal{F}$, there exists a sequence of functions in $\{h_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$ such that $\lim_{n\to\infty} h_n(x) = f(x)$ for all $x \geq 0$ and $h_n(x) \leq f(x) \leq x$ for all $x \geq 0$ and $n = 1, 2, \ldots$.*

Proof. For any non-negative increasing convex function f defined on $[0, \infty)$, there exists a sequence of non-negative functions $\{h_n, n = 1, 2, \ldots\}$ defined on $[0, \infty)$ such that:

$$h_n(x) = \sum_{j=1}^{n} c_{n,j}(x - d_{n,j})_+$$

for constants $c_{n,j} \geq 0$, $d_{n,j} \geq 0$ and $\lim_{n\to\infty} h_n(x) = f(x)$ for all $x \geq 0$.

If $\lim_{n\to\infty} h_n(x) = f(x)$ for all $x \geq 0$, then $h_n(x) \leq f(x)$. Moreover, for any f which is an element of $\mathcal{F}$, we have $0 \leq f(x) \leq x$ which implies that $h_n(x) \leq f(x) \leq x$ for all $x \geq 0$ and $n = 1, 2, \ldots$. □

Now let h^* be any optimal ceded loss function in class $\mathcal{H}$ under VaR, which implies that the

$$\mathrm{VaR}_{T_{h_n}(X)}(\alpha) \geq \mathrm{VaR}_{T_{h^*}(X)}(\alpha) \quad \text{for any} \quad n = 1, 2, \ldots$$

We need to demonstrate that the $\mathrm{VaR}_{T_f(X)}(\alpha) \geq \mathrm{VaR}_{T_{h^*}(X)}(\alpha)$ for any $f \in \mathcal{F}$. From Lemma 3.1 we have $\{h_n, n = 1, 2, \ldots\}$ in $\mathcal{H}$ such that $\lim_{n\to\infty} h_n(x) = f(x)$ for all $x \geq 0$ and $h_n(x) \leq f(x) \leq x$ for all $x \geq 0$ and $n = 1, 2, \ldots$.

The dominated convergence theorem states that if f_n is a sequence of measurable functions, f_n approaches f pointwise as n approaches ∞, and $|f_n| \leq f$ where f is integrable, then $\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu$.

This implies that $\lim_{n\to\infty} \mathbb{E}[h_n(X)] = \mathbb{E}[f(X)]$. We now see that $T_{h_n}(X) = I_{h_n}(X) + \delta_{h_n}(X) = I_{h_n}(X) + (1+\rho)\mathbb{E}[h_n(X)] \to I_f(X) + (1+\rho)\mathbb{E}[f(X)]$ as n approaches ∞.

If functions f_n and f are increasing and continuous and satisfy $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \geq 0$, then $\lim_{n\to\infty} \mathrm{VaR}_{f_n(X)}(\alpha) = \mathrm{VaR}_{f(X)}(\alpha)$. Thus, $\lim_{n\to\infty} \mathrm{VaR}_{T_{h_n}(X)}(\alpha) = \mathrm{VaR}_{T_f(X)}(\alpha)$ and $\mathrm{VaR}_{T_f(X)}(\alpha) \geq \mathrm{VaR}_{T_{h^*}(X)}(\alpha)$. Together this brings the following conclusion:

Lemma 3.2. *The optimal ceded loss function which minimizes the VaR of the insurer's total risk in class $\mathcal{H}$ is also optimal in the class $\mathcal{F}$.*

We then define the following sets:

$$\begin{cases} D_n = \{(d_{n,1}, \ldots, d_{n,n}) : 0 \leq d_{n,1} \leq \cdots \leq d_{n,n}\}, \\ D_n^0 = \{(d_{n,1}, \ldots, d_{n,n}) : S_X^{-1}(\alpha) \leq d_{n,1} \leq \cdots \leq d_{n,n}\}, \\ D_n^i = \{(d_{n,1}, \ldots, d_{n,n}) : 0 \leq d_{n,1} \leq \cdots \leq d_{n,i} \leq S_X^{-1}(\alpha) \leq d_{n,i+1} \leq \cdots \leq d_{n,n}, \ i = 1, \ldots, n-1\}, \\ D_n^n = \{(d_{n,1}, \ldots, d_{n,n}) : 0 \leq d_{n,1} \leq \cdots \leq d_{n,n} \leq S_X^{-1}(\alpha)\}. \end{cases}$$

Lemma 3.3. *For any $h(x) = \sum_{j=1}^{n} c_{n,j}(x - d_{n,j})_+ \in \mathcal{H}$ and a given confidence interval $1-\alpha$ with $0 < \alpha < S_X(0)$, we have 3 different scenarios for $VaR_{T_h(X)}(\alpha)$:*

$$VaR_{T_h(X)}(\alpha) = \begin{cases} S_X^{-1}(\alpha) + \delta_h(X), & S_X^{-1}(\alpha) \leq d_{n,1} \\ A_{n,i}S_X^{-1}(\alpha) + B_{n,i} + \delta_h(X) & d_{n,i} \leq S_X^{-1}(\alpha) \leq d_{n,i+1}, i = 1, \ldots, n-1 \\ A_{n,n}S_X^{-1}(\alpha) + B_{n,n} + \delta_h(X) & d_{n,n} \leq S_X^{-1}(\alpha) \end{cases}$$

1. *When $S_X^{-1}(\alpha) \leq d_{n,1}$, under D_n^0,*

$$VaR_{T_h(X)}(d_{n,1}, \ldots, d_{n,n}, \alpha) = S_X^{-1}(\alpha) + \frac{1}{\rho^*}\left\{\sum_{j=1}^{n} c_{n,j} \int_{d_{n,j}}^{\infty} S_X(x) \, dx\right\} \tag{3.1.5}$$

2. *When $d_{n,i} \leq S_X^{-1}(\alpha) \leq d_{n,i+1}$, under D_n^i, for $i = 1, \ldots, n-1$,*

$$VaR_{T_h(X)}(d_{n,1}, \ldots, d_{n,n}, \alpha) =$$

$$\left(1-\sum_{j=1}^{i} c_{n,j}\right) S_X^{-1}(\alpha)+\sum_{j=1}^{i} c_{n,j} g(d_{n,j})+\frac{1}{\rho^*}\left\{\sum_{j=i+1}^{n} c_{n,j} \int_{d_{n,j}}^{\infty} S_X(x)\, dx\right\}=$$

$$\left(1-\sum_{j=1}^{i} c_{n,j}\right) S_X^{-1}(\alpha)+\sum_{j=1}^{i} c_{n,j} d_{n,j}+\frac{1}{\rho^*}\left\{\sum_{j=1}^{n} c_{n,j} \int_{d_{n,j}}^{\infty} S_X(x)\, dx\right\} \tag{3.1.6}$$

3. *When $d_{n,n} \leq S_X^{-1}(\alpha)$, under D_n^n,*

$$VaR_{T_h(X)}(d_{n,1},\ldots,d_{n,n},\alpha)=\left(1-\sum_{j=1}^{n} c_{n,j}\right) S_X^{-1}(\alpha)+\sum_{j=1}^{n} c_{n,j} g(d_{n,j})=$$

$$\left(1-\sum_{j=1}^{n} c_{n,j}\right) S_X^{-1}(\alpha)+\sum_{j=1}^{n} c_{n,j} d_{n,j}+\frac{1}{\rho^*}\left\{\sum_{j=1}^{n} c_{n,j} \int_{d_{n,j}}^{\infty} S_X(x)\, dx\right\} \tag{3.1.7}$$

To obtain Lemma 3.4, we will analyze the minimum of $\mathrm{VaR}_{T_h(X)}(d_{n,1},\ldots,d_{n,n},\alpha)$ on the set D_n for $n=1,2\ldots$ by discussing the infimum on D_n^i for each $i=0,1,2,\ldots,n$. Hence, the goal is to verify that $\mathrm{VaR}_{T_h(X)}(d_{n,1},\ldots,d_{n,n},\alpha)$ has a minimum on D_n if and only if:

$$\min_{D_n^n}\left\{\mathrm{VaR}_{T_h(X)}(d_{n,1},\ldots,d_{n,n},\alpha)\right\} \leq \min_{i=0,1,\ldots,n-1}\left\{\inf_{D_n^i} \mathrm{VaR}_{T_h(X)}(d_{n,1},\ldots,d_{n,n},\alpha)\right\}$$

We consider the value of the infimum on the set D_n^i for $i=0,1,2,\ldots,n$. Note that $\int_d^\infty S_X(x)\, dx$ is a decreasing function when d is greater than 0, and $\int_d^\infty S_X(x)\, dx \to 0$ as $d\to\infty$.

First, we look at the relationship from 3.1.1 under D_n^0, which states:

$$\mathrm{VaR}_{T_h(X)}(\infty,\ldots,\infty,\alpha)=S_X^{-1}(\alpha)+\frac{1}{\rho^*}\left\{\sum_{j=1}^{n} c_{n,j} \int_{d_{n,j}}^{\infty} S_X(x)\, dx\right\} \text{ on } D_n^0$$

Since $\int_{d_{n,j}}^\infty S_X(x)\, dx \to 0$, we are then left with the simplified equation:

$$\mathrm{VaR}_{T_h(X)}=S_X^{-1}(\alpha) \text{ on } D_n^0$$

We must consider three cases for the set D_n^i:

1. Recall (3.1.6) under D_n^i:

$$\mathrm{VaR}_{T_h(X)}=\left(1-\sum_{j=1}^{i} c_{n,j}\right) S_X^{-1}(\alpha)+\sum_{j=1}^{i} c_{n,j} g(d_{n,j})+\frac{1}{\rho^*}\left\{\sum_{j=i+1}^{n} c_{n,j} \int_{d_{n,j}}^{\infty} S_X(x)\, dx\right\}$$

Again, $\int_{d_{n,j}}^\infty S_X(x)\, dx \to 0$, so:

$$\mathrm{VaR}_{T_h(X)}=\left(1-\sum_{j=1}^{i} c_{n,j}\right) S_X^{-1}(\alpha)+\sum_{j=1}^{i} c_{n,j} g(d_{n,j})$$

If $\rho^* < S_X(0)$, then the continuous function $g(x)$ is decreasing on $(0,d^*)$ while increasing on (d^*,∞) and satisfies $\min_{0\leq x\leq a}\{g(x)\}=g(d^*)=u(\rho^*)$ for $d^*\leq a$. When $\rho^*<S_X(0)$, $d^*=S_X^{-1}(\rho^*)\leq S_X^{-1}(\alpha)$ and $i=1,\ldots,n-1$:

$$\inf_{D_n^i}\left\{\mathrm{VaR}_{T_h(X)}\right\}=\mathrm{VaR}_{T_h(X)}(d^*,\ldots,d^*,\infty,\ldots,\infty,\alpha)=\left(1-\sum_{j=1}^{i} c_{n,j}\right) S_X^{-1}(\alpha)+\sum_{j=1}^{i} c_{n,j} g(d^*)$$

When we distribute the inverse survival function and replace $g(d^*)$ with $u(\rho^*)$, we get the following relationship:

$$= S_X^{-1}(\alpha) - S_X^{-1}(\alpha)\sum_{j=1}^{i} c_{n,j} + \sum_{j=1}^{i} c_{n,j} u(\rho^*)$$

Finally, once we rearrange the above formula, we get:

$$= S_X^{-1}(\alpha) + \left[u(\rho^*) - S_X^{-1}(\alpha)\right]\sum_{j=1}^{i} c_{n,j} \text{ on } D_n^i$$

2. If $\rho^* < S_X(0)$, then the continuous function $g(x)$ is decreasing on $(0, d^*)$ while increasing on (d^*, ∞) and satisfies $\min_{0\le x\le a}\{g(x)\} = g(a)$ for $0 \le a \le d^*$. When $\rho^* < S_X(0)$ and $d^* = S_X^{-1}(\rho^*) > S_X^{-1}(\alpha)$; when $i = 1, \ldots, n-1$, we can follow similar logic as above to achieve the formula:

$$\begin{aligned} &\mathrm{VaR}_{T_h(X)}(S_X^{-1}(\alpha), \ldots, S_X^{-1}(\alpha), \infty, \ldots, \infty, \alpha) \\ &= \left(1 - \sum_{j=1}^{i} c_{n,j}\right) S_X^{-1}(\alpha) + \sum_{j=1}^{i} c_{n,j} g(S_X^{-1}(\alpha)) \\ &= S_X^{-1}(\alpha) + \left[u(\alpha) - S_X^{-1}(\alpha)\right]\sum_{j=1}^{i} c_{n,j} \end{aligned}$$

3. When $\rho^* \ge S_X(0)$; when $i = 1, \ldots, n-1$, we can follow similar logic as above to achieve the formula:

$$\begin{aligned} &\mathrm{VaR}_{T_h(X)}(0, \ldots, 0, \infty, \ldots, \infty, \alpha) \\ &= \left(1 - \sum_{j=1}^{i} c_{n,j}\right) S_X^{-1}(\alpha) + \sum_{j=1}^{i} c_{n,j} g(0) \\ &= S_X^{-1}(\alpha) + \left[g(0) - S_X^{-1}(\alpha)\right]\sum_{j=1}^{i} c_{n,j} \end{aligned}$$

Combining (1), (2), and (3), we can see that for any function $h(x)$ with fixed coefficients, its VaR has a minimum only on the set D_n^n and has only an infimum on all other sets $D_n^0, D_n^1, \ldots D_n^{n-1}$. This statement implies that the VaR of the ceded loss function has a minimum on the set D_n if and only if:

$$\min_{D_n^n}\left\{\mathrm{VaR}_{T_h(X)}(d_{n,1}, \ldots, d_{n,n}, \alpha)\right\} \le \min_{i=0,1,\ldots,n-1}\left\{\inf_{D_n^i} \mathrm{VaR}_{T_h(X)}(d_{n,1}, \ldots, d_{n,n}, \alpha)\right\}$$

This relationship is what we sought to verify initially. Note that when this inequality holds, we get the relationship:

1. When $u(\rho^*) - S_X^{-1}(\alpha) \le 0$,

$$\min_{D_n}\left\{\mathrm{VaR}_{T_h(X)}(d_{n,1}, \ldots, d_{n,n}, \alpha)\right\} = \min_{D_n^n}\left\{\mathrm{VaR}_{T_h(X)}(d_{n,1}, \ldots, d_{n,n}, \alpha)\right\}$$

$$= \mathrm{VaR}_{T_h(X)}(d^*, \ldots, d^*, \alpha) = S_X^{-1}(\alpha) + \left[u(\rho^*) - S_X^{-1}(\alpha)\right]\sum_{j=1}^{n} c_{n,j}$$

2. When $g(0) - S_X^{-1}(\alpha) \le 0$,

$$\min_{D_n}\left\{\mathrm{VaR}_{T_h(X)}(d_{n,1}, \ldots, d_{n,n}, \alpha)\right\} = \min_{D_n^n}\left\{\mathrm{VaR}_{T_h(X)}(d_{n,1}, \ldots, d_{n,n}, \alpha)\right\}$$

$$= \mathrm{VaR}_{T_h(X)}(0, \ldots, 0, \alpha) = S_X^{-1}(\alpha) + \left[g(0) - S_X^{-1}(\alpha)\right]\sum_{j=1}^{n} c_{n,j}$$

Lemma 3.4. *Given a confidence level $1-\alpha$ with $0<\alpha<S_X(0)$, for any function $h(x)$:*

1. *If $\rho^* < S_X(0)$ and $S_X^{-1}(\alpha) \geq u(\rho^*)$:*

$$\min_{D_n}\left\{ VaR_{T_h(X)}(d_{n,1},\ldots,d_{n,n},\alpha)\right\} = S_X^{-1}(\alpha) + \left[u(\rho^*) - S_X^{-1}(\alpha)\right]\sum_{j=1}^{n} c_{n,j}$$

where the minimum VaR is obtained at:

$$h^*(x) = \sum_{j=1}^{n} c_{n,j}(x-d^*)$$

2. *If $\rho^* \geq S_X(0)$ and $S_X^{-1}(\alpha) \geq g(0)$:*

$$\min_{D_n}\left\{ VaR_{T_h(X)}(d_{n,1},\ldots,d_{n,n},\alpha)\right\} = S_X^{-1}(\alpha) + \left[g(0) - S_X^{-1}(\alpha)\right]\sum_{j=1}^{n} c_{n,j}$$

where the minimum VaR is obtained at:

$$h^*(x) = \sum_{j=1}^{n} c_{n,j}(x)$$

3. *For any other cases, minimum VaR does not exist.*

Utilizing all of the Lemmas previously discussed, we can now summarize a Theorem for practical reinsurance applications.

Theorem 3.5. *For a given confidence level $1-\alpha$ with $0<\alpha<S_X(0)$:*

1. *If $\rho^* < S_X(0)$ and $S_X^{-1}(\alpha) > u(\rho^*)$, then the minimum VaR is $u(\rho^*)$, and is achieved at:*

$$f^*(x) = (x-d^*)_+$$

where d^* is defined on page 6.

2. *If $\rho^* < S_X(0)$ and $S_X^{-1}(\alpha) = u(\rho^*)$, then the minimum VaR is $S_X^{-1}(\alpha)$, and is achieved at:*

$$f^*(x) = c(x-d^*)_+$$

for any constant $0<c\leq 1$.

3. *If $\rho^* \geq S_X(0)$ and $S_X^{-1}(\alpha) > g(0)$, then the minimum VaR is $g(0)$, and is achieved at:*

$$f^*(x) = x$$

4. *If $\rho^* \geq S_X(0)$ and $S_X^{-1}(\alpha) = g(0)$, then the minimum VaR is $S_X^{-1}(\alpha)$, and is achieved at:*

$$f^*(x) = cx$$

for any constant $0<c\leq 1$.

3.2 VaR and CTE - a Geometric Approach

To begin, we use the following definition:

$$H(f) = \mathrm{VaR}_{T_f(X)}(\alpha) \tag{3.2.1}$$

Using the translation invariance principle from 2.2.3, we can effectively reach the following result:

$$\begin{aligned} H(f) &= \mathrm{VaR}_{I_f(X)}(\alpha) + (1+\rho)\mathbb{E}[f(X)] \\ &= I_f(\mathrm{VaR}_X(\alpha)) + (1+\rho)\mathbb{E}[f(X)] \\ &= S_X^{-1}(\alpha) - f(S_X^{-1}(\alpha)) + (1+\rho)\mathbb{E}[f(X)] \end{aligned}$$

When we replace $S_X^{-1}(\alpha)$ with a we can reach the following optimization problem:

$$\min_{f\in\mathcal{F}} H(f) = \min_{f\in\mathcal{F}}\{a - f(a) + (1+\rho)\mathbb{E}[f(X)]\} \tag{3.2.2}$$

Lemma 3.6. *A ceded loss function f in $\mathcal{F}$ that is not null but identically zero on $[0, a]$ is not optimal for Equation 3.2.2.*

Proof. Let f be such a function. Consider $h = \frac{1}{2}f \in \mathcal{F}$. Since $H(f) = a + (1+\rho)\mathbb{E}[f(X)] > a + (1+\rho)\mathbb{E}[h(X)] = H(h)$, f is not optimal. $\square$

Therefore, we must assume that $\mathcal{F}$ does not contain any non-null functions that are identically zero on $[0, a]$.

Lemma 3.7. *Let $f \in \mathcal{F}$ be a non-null ceded loss function. There always exists a function $h \in \mathcal{G}$ such that $H(h) \leq H(f)$.*

Proof. Let $\mathcal{G} \subset \mathcal{F}$ be the collection of functions of form:

$$f_{c,d}(x) = c(x-d)_+$$

where $(c, d) \in [0, 1] \times [0, a)$ and which includes $f_{c,0}(x) = cx$ and the null function $f_{0,d} \equiv 0$.

A supporting line of slope c, and which passes through the point $(a, f(a))$ always lies below the convex function $f(x)$. Since $0 \leq f(x) \leq x$ for all $f(x)$ not excluded from Lemma 3.6, c must be between 0 and 1.

We define d as the intersection of the supporting line and the x-axis. Thus, $d = a - \frac{f(a)}{c}$. When we define $h(x) = c(x-d)_+$, then:

$$H(h) = a - h(a) + (1+\rho)\mathbb{E}[h(X)] \leq a - f(a) + (1+\rho)\mathbb{E}[f(X)] = H(f)$$

since $h(a) = f(a)$ and $\mathbb{E}[h(X)] \leq \mathbb{E}[f(X)]$. $\square$

From this, we can achieve a similar theorem as in Theorem 3.5

Theorem 3.8. *For a given confidence level $\alpha \in (0, S_X(0))$ the following statements hold true:*

1. *If $\rho^* < S_X(0)$ and $a > u(\rho^*)$, then the minimum value of H over $\mathcal{F}$ is $g(d^*)$, and the optimal ceded loss function is:*
$$f^*(x) = (x-d^*)_+$$
where $g(d^*) = u(\rho^*)$.

2. *If $\rho^* < S_X(0)$ and $a = u(\rho^*)$, then the minimum value of H over $\mathcal{F}$ is $g(d^*)$, and the optimal ceded loss function is:*
$$f^*(x) = c(x-d^*)_+$$
for any constant $c \in [0, 1]$.

3. *If $\rho^* \geq S_X(0)$ and $a > g(0)$, then the minimum value of H over $\mathcal{F}$ is $g(0)$, and the optimal ceded loss function is:*
$$f^*(x) = x$$

4. *If $\rho^* \geq S_X(0)$ and $a = g(0)$, then the minimum value of H over $\mathcal{F}$ is $g(0)$, and the optimal ceded loss function is:*
$$f^*(x) = cx$$
for any constant $c \in [0, 1]$.

5. *For all other cases, the minimum value of H over $\mathcal{F}$ is a, and the optimal ceded loss function is $f^*(x) \equiv 0$.*

3.3 VaR Numerical Analysis

As mentioned, the goal of this research project is to conduct a numerical study to practically implement the theory in the reinsurance market. To accomplish this, we picked a distribution to model the loss X for a sample insurance company. In this case, we chose the exponential distribution with the survival function $S(x) = (1-z)e^{-\lambda x}$, where $0 \leq z < 1$. From here, we just follow the guidelines set by Theorem 3.5.

In short, here are all the formulas and calculations that were required:

$$S(x) = (1-z)e^{-\lambda x} \qquad S^{-1}(x) = \frac{-\ln\left(\frac{x}{1-z}\right)}{\lambda}$$

$$\rho^* = \frac{1}{1+\rho} \qquad S(0) = 1-z$$

$$d^* = \frac{-\ln\left(\frac{\rho^*}{1-z}\right)}{\lambda} \qquad S^{-1}(\alpha) = \frac{-\ln\left(\frac{\alpha}{1-z}\right)}{\lambda}$$

$$u(\rho^*) = S_X^{-1}(\rho^*) + \frac{1}{\rho^*}\int_{S_X^{-1}(\rho^*)}^{\infty} S_X(t)\,dt = \frac{-\ln\left(\frac{\rho^*}{1-z}\right)}{\lambda} + \frac{1}{\lambda}$$

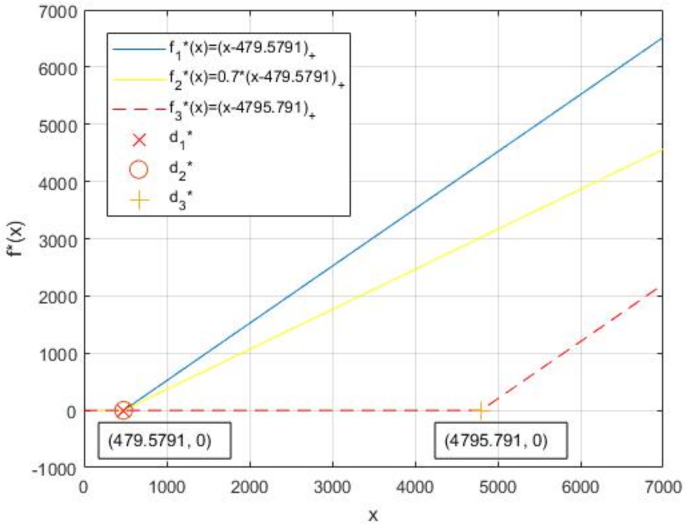
$$g(0) = 0 + \frac{1}{\rho^*}\int_0^{\infty} S_X(t)\,dt = \frac{1-z}{\rho^*\lambda}$$

From here, it's just a matter of plugging in various values for ρ, α, z and λ. ρ, the safety loading, is a form of protection against insolvency set by the company executives. α, the confidence, is a value set by industry regulators which affects the amount of risk an insurer will take. For now we set z to be 0. z is the "jump" of the survival function. In the real world, when z is greater than 0, it represents the possibility that there are no claims made to the insurance company by its policyholders. λ is just a parameter of the exponential distribution.

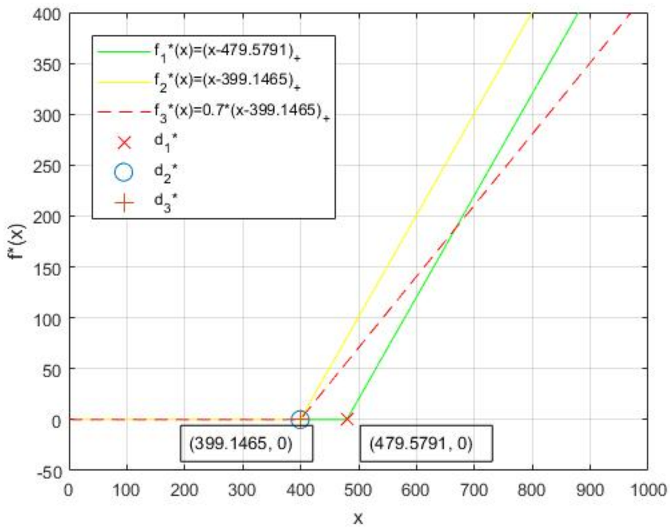
For example, if ρ is 10, $\alpha = .01$, and $\lambda = .005$, we see that ρ^* is less than $S_X(0)$ and that $S_X^{-1}(\alpha)$ is greater than $u(\rho^*)$ as described under part (1) of Theorem 3.5. Thus, under these criteria, we see that the optimal reinsurance contract is a stop-loss contract.

Here is a table of the values that we experimented with:

ρ	α	λ	z	VaR	f^*
10	.01	.005	0	Stop Loss	$(x-479.579)_+$
10	.0334	.005	0	Change Loss	$c(x-479.579)_+$
10	.05	.005	0	No Solution	
6.36	.05	.005	0	Change Loss	$c(x-399.146)_+$
6.36	.025	.005	0	Stop Loss	$(x-399.146)_+$
10	.01	.0005	0	Stop Loss	$(x-4795.79)_+$



From here we can see some interesting results. For example, we see that increasing α changes our solution from a stop-loss to a change-loss and eventually to no solution. In real terms, this means as α increases, the insurer is willing to hold on to more and more risk; eventually, the insurer will stop buying reinsurance altogether.

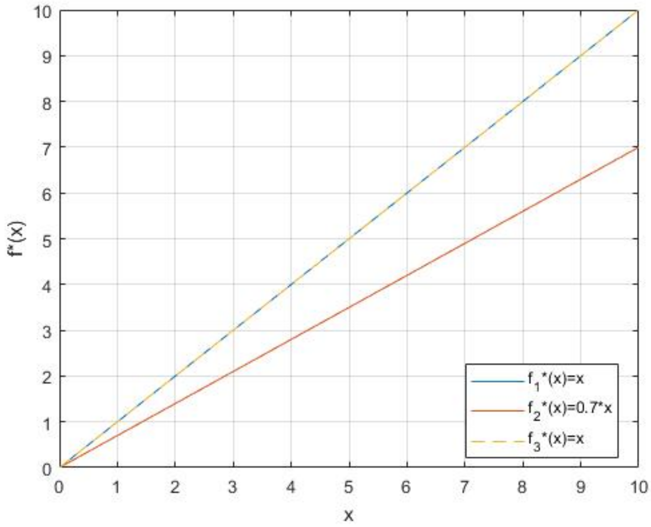


We can also see that decreasing ρ not only decreases the deductible but increases the largest possible α which can provide a solution.

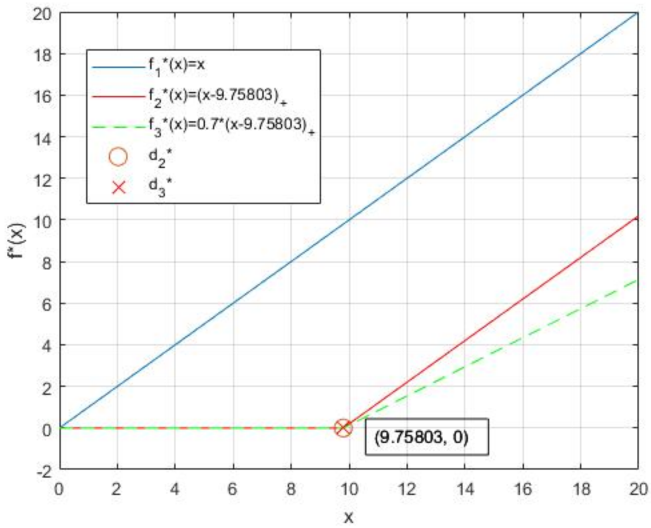
To obtain part (3) and (4) of Theorem 3.5, we require that $\rho^* < S_X(0)$. This is not possible while $z = 0$. So we just set $z = .95$ and this results in the following table:

ρ	α	λ	z	VaR	f^*
10	.01	.005	.95	Quota Share	x
10	.0288	.005	.95	Quota Share	cx
10	.05	.005	.95	No Solution	
20	.01	.005	.95	Stop Loss	$(x-9.758)_+$
20	.0175	.005	.95	Change Loss	$c(x-9.758)_+$
10	.01	.0005	.95	Quota Share	x

Since $z = .95$, this means that there is a possibility of having no claims.



We see some parallels from our previous results. Again, α converts our solution from quota share to no solution. Finally, λ has absolutely no effect, since the deductible does not apply for the quota share contract.



Increasing ρ changes our solution back to our original stop-loss/change-loss scenario from before.

3.4 Optimal Reinsurance under CTE Risk Measure

Following similar arguments as under VaR, we can obtain a theorem for optimal reinsurance under CTE.

Theorem 3.9. *For a given confidence interval $1-\alpha$ with $0<\alpha<S_X(0)$:*

1. *If $\alpha<\rho^*<S_X(0)$, then $min_{f\in F}\ CTE_{T_f(X)}(\alpha)=u(\rho^*)$ and the minimum CTE is attained at:*
$$f^*(x)=(x-d^*)_+$$
where f^* is the optimal ceded loss function.

2. *If $\alpha=\rho^*<S_X(0)$, then $min_{f\in F}\ CTE_{T_f(X)}(\alpha)=u(\rho^*)$ and the minimum CTE is attained at:*
$$f^*(x)=\sum_{j=1}^{n}c_{n,j}(x-d_{n,j})_+\in\mathcal{H}$$
such that $d^*\leq d_{n,1}\leq\cdots\leq d_{n,n}$ and $n=1,2,\ldots$.

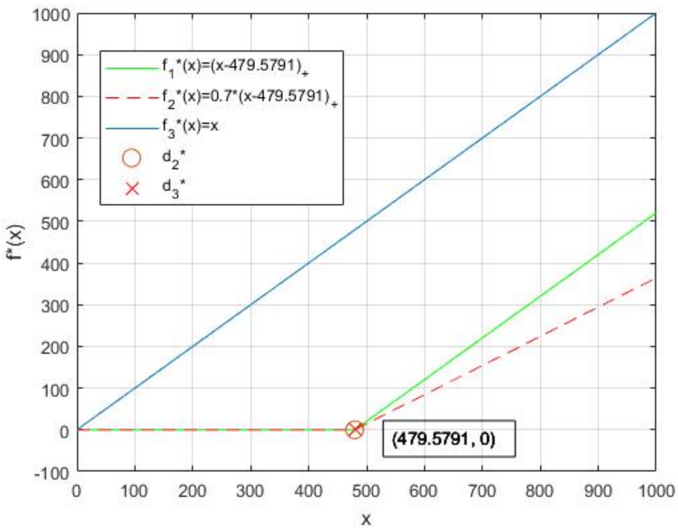
3. *If $\alpha < S_X(0) \leq \rho^*$, then $min_{f\in F}$ $CTE_{T_f(X)}(\alpha) = u(\rho^*)$ and the minimum CTE is attained at:*

$$f^*(x) = x$$

3.5 CTE Numerical Analysis

Following similar studies as under the VaR numerical study, we can obtain a table and graph for optimal reinsurance under CTE:

ρ	α	λ	z	VaR	f^*
10	.01	.005	0	Stop Loss	$(x - 479.579)_+$
10	.0909	.005	0	Change Loss	$c(x - 479.579)_+$
10	.01	.005	.95	Quota Share	x



Again, we see that increasing α changes our solution from a stop-loss to a change-loss. Again, this means as α increases, the insurer is willing to hold on to more and more risk. Finally, we see again that increasing z causes the insurer to become more risk adverse.

4 Optimal Reinsurance under VaR with Wang's Premium Principle

4.1 Wang's Premium Principle

Instead of using the expectation premium principle, $(1+\rho)E[f(X)]$, we may also use other methods for determining the premium under our VaR and CTE objective functions. One of these principles is Wang's premium principle.

We begin by defining Wang's premium principle:

$$H_w(X) = \int_0^{\infty} w(S_X(t)))dt$$

where the distortion function w is a non-decreasing, concave function such that $w(0) = 0$ and $w(1) = 1$.

By replacing the expectation premium from the previous section with Wang's Premium Principle, we get:

$$T_f(X) = I_f(X) + H_w(f(X))$$

Using $B(f) = \mathrm{VaR}_{T_f(X)}(\alpha)$, we get:

$$\min_{f\in\mathcal{F}} B(f) = \min_{f\in\mathcal{F}} \{a - f(a) + H_w(f(X))\} \tag{4.1.1}$$

Theorem 4.1. *For a given confidence level $\alpha \in (0, S_X(0))$ the following statements hold true:*

1. *If $H_w(X) < a$, then the minimum value of B over $\mathcal{F}$ is $H_w(X)$, and the optimal ceded loss function is*
$$f^*(x) = x$$
2. *If $H_w(X) = a$, then the minimum value of B over $\mathcal{F}$ is $H_w(X)$, and the optimal ceded loss function is*
$$f^*(x) = cx$$
for any constant $c \in (0, 1]$.
3. *If $H_w(X) > a$, then the minimum value of B over $\mathcal{F}$ is a, and the optimal ceded loss function is*
$$f^*(x) \equiv 0$$

4.2 Wang's Premium Principle Numerical Analysis

In this numerical study, we continue to use the exponential function as the distribution function for the non-negative random variable X, this time with the survival function $S(x) = e^{\frac{-x}{\theta}}$.

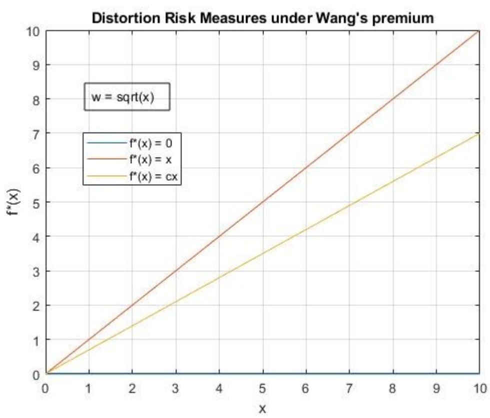
In short, here are all the equations that were required:

$$S^{-1}(x) = -\theta \ln(x)$$
$$a = \mathrm{VaR}(\alpha) = S^{-1}(x) = -\theta \ln(\alpha)$$
$$w = \sqrt{x} \quad x \in [0, 1]$$
$$H_w(t) = \int_0^\infty \sqrt{e^{-\frac{t}{\theta}}}\, dt = 2\theta$$

There are two major reasons that we selected $w = \sqrt{x}$ as the distortion function. First, an ideal premium principle should be no less than its expectation $E(X) = x$. In this case, for $x \in [0, 1]$, $\sqrt{x} \geq x$. Secondly, the selected distortion function $\sqrt{x}$ exaggerates big losses and reduces the weight of small losses, which is considered to be a desirable feature.

From here, we set $\theta = 0.005$, and hence $H_w(X) = 0.01$. We consider various values for α under different scenarios and calculate the optimal function in each scenario. The results are shown in the following table.

θ	$H_w(X)$	α	a	f^*
0.005	.01	.01	0.0230	x
0.005	.01	.1353	0.01	cx
0.005	.01	.15	.0095	0



5 The Cost-Benefit Approach

5.1 Free-premium Problem

We can also use other objective functions other than VaR and CTE. To accomplish this, we look at the minimization of an insurer's risk-adjusted liability, which is composed of:

1. The actuarial reserve on the insurer's total retained risk, denoted by:
$$\mathbb{E}[T_f(X)]$$
2. A risk margin which is denoted by:
$$\text{Risk margin} = \delta\rho_g(T_f(X) - \mathbb{E}[T_f(X)])$$
where δ is the cost-of-capital rate, known more commonly as the interest rate. The risk margin acts as a buffer, and is added to the actuarial reserve. This covers for any adverse deviation of risk from its normally expected value.

We now have both parts to give the risk-adjusted liability:

$$L_f(X) = \mathbb{E}[T_f(X)] + \delta\rho_g(T_f(X) - \mathbb{E}[T_f(X)]) \tag{5.1.1}$$

Using the translation invariance property, we can rewrite the risk adjusted liability as follows:

$$\begin{aligned} L_f(X) &= \mathbb{E}[T_f(X)] + \delta\rho_g(T_f(X) - \mathbb{E}[T_f(X)]) \\ L_f(X) &= \mathbb{E}[T_f(X)] + \delta\rho_g(T_f(X)) - \delta\mathbb{E}[T_f(X)] \\ L_f(X) &= (1-\delta)\mathbb{E}[T_f(X)] + \delta\rho_g(T_f(X)) \end{aligned}$$

In this form, we can view the risk-adjusted liability as a weighted average of the actuarial reserve on the distortion risk measure of the total retained risk. Note that when $\delta = 1, L_f(X) = \rho_g(T_f(X))$.

We assume a general reinsurance premium for a given ceded loss function as:

$$\mu_r(f(X)) = \int_0^\infty r(S_{f(X)}(t))\ dt$$

where the function $r : [0, 1]$ in the set of positive real numbers is a non-decreasing function where $r(0) =\ 0$. For simplicity sake, we assume that r is not zero almost everywhere.

There are two optimization problems to consider:

1. Free-premium problem:
$$\inf_{f\in F} L_f(X)$$
2. Budget-constrained problem:
$$\begin{cases} inf_{f\in F}, & L_f(X) \\ s.t., & \mu_r(f(X)) \leq \pi \end{cases}$$
where π is the budget allocated to reinsurance, typically set by an insurance company executive team.

Using the indicator function, we see that:

$$\mathbb{E}[f(X)] = \int_0^\infty \mathbb{E}\left[\mathbb{1}_{\{X>t\}}\right]\ df(t) = \int_0^\infty S_X(t)\ df(t)$$

To move forward, we must prove the following relationship:

$$\rho_g(f(X)) = \int_0^\infty g(S_X(t))\ df(t)$$

To begin:

$$F^{-1}_{f(X)}(1-p) = \int_0^\infty F^{-1}_{\mathbb{1}_{\{X>t\}}}(1-p)\ df(t), \quad 0<p<1$$

We can use $F^{-1}_{f(X)}(p) = f(F^{-1}_X(p))$ and $F^{-1}(p) \leq x = p \leq F(x)$ to arrive at:

$$F^{-1}_{\mathbb{1}_{\{x>t\}}}(1-p) = \begin{cases} 0, & 0 < 1-p \leq F_X(t) \\ 1, & F_X(t) < 1-p \leq 1 \end{cases}$$
$$= \begin{cases} 0, & t \geq F^{-1}_X(1-p) \\ 1, & t < F^{-1}_X(1-p) \end{cases}$$

Hence:

$$\int_0^\infty F^{-1}_{\mathbb{1}_{\{X>t\}}}(1-p)\ df(t) = \int_0^{F_x^{-1}(1-p)} df(t) = f(F^{-1}_X(1-p))$$

This verifies the left-continuous case. The right continuous case can be verified in a similar manner. To prove $\rho_g(f(X))$, we consider three cases.

1. g is left-continuous

$$\rho_g(X) = \int_0^1 F^{-1}_X(1-p)\ dg(p)$$
$$\rho_g(f(X)) = \int_0^1 F^{-1}_{f(X)}(1-p)\ dg(p)$$
$$= \int_0^1 \left(\int_0^\infty F^{-1}_{\mathbb{1}_{\{X>t\}}}(1-p)\ df(t)\right) dg(p)$$
$$= \int_0^\infty \left(\int_0^1 F^{-1}_{1_{\{X>t\}}}(1-p)\ dg(p)\right) df(t)$$
$$= \int_0^\infty \left(\int_0^{S_X(t)} dg(p)\right) df(t)$$
$$= \int_0^\infty g(S_X(t))\ df(t)$$

2. g is right-continuous, which can be solved in a similar manner as above.

3. For a general g, we write $g = c_r g_r + c_l g_l$ for some right-continuous distortion function g_r and some left-continuous distortion function g_l, as well as non-negative constants c_r and c_l such that $c_r + c_l = 1$. It follows then from Case 1 and Case 2 that:

$$\rho_g(f(X)) = c_r \rho_{g_r}(f(X)) + \ c_l \rho_{g_l}(f(X))$$

$$= \int_0^\infty g(S_X(t))\ df(t)$$

In a similar fashion, we can also see that:

$$\mu_r(f(X)) = \int_0^\infty r(S_X(t))\ df(t)$$

Using the formulas $L_f(X) = (1-\delta)\mathbb{E}[T_f(x)] + \delta\rho_g(T_f(X))$ and $\mathbb{E}[f(X)] = \int_0^\infty S_X(t)\ df(t)$ we have that:

$$L_f(X) = (1-\delta)\mathbb{E}[T_f(X)] + \delta\rho_g(T_f(X))$$
$$= (1-\delta)\mathbb{E}[X - f(X) + \ \mu_r(f(X))] + \delta\rho_g(T_f(X))$$
$$= (1-\delta)\mathbb{E}[X] - \mathbb{E}[f(X)] + \mu_r(f(X)) + \delta\rho_g(T_f(X))$$

$$= (1-\delta)[\mathbb{E}[X] + \int_0^\infty [r(S_X(t)) - S_X(t)]\ df(t)] + \delta[\rho_g(X) + \int_0^\infty r(S_X(t)) - g(S_X(t))\ df(t)]$$

$$= \delta\rho_g(X) + (1-\delta)\mathbb{E}[X] + \int_0^\infty G_X(t)\ df(t)$$

Here, $G_X(t)$ consists of:

$$G_X(t) = r(S_X(t)) - [\delta g(S_X(t)) + (1-\delta)S_X(t)]$$

Lemma 5.1.

$$\mathbb{E}[f(X)] = \int_0^\infty S_X(t)\ df(t)$$

In this context, we can express $L_f(X)$ as:

$$L_f(X) = \delta\rho_g(X) + (1-\delta)\mathbb{E}[X] + \int_0^\infty G_X(t)\ df(t)$$

$$= \delta\rho_g(X) + (1-\delta)\mathbb{E}[X] + \int_0^\infty G_X(t)\ f^{'}(t)dt$$

Additionally, the premium can be represented as:

$$\mu_r(f(X)) = \int_0^\infty r(S_X(t))\ df(t) = \int_0^\infty r(S_X(t))f^{'}(t)\ dt$$

Using this the above lemma, we can stipulate the following theorem.

Theorem 5.2. *Every optimal solution is of the form:*

$$f^{*1}(x) = \int_0^x 1_{\{G_X<0\}}(t)\ dt + \int_0^x 1_{\{G_X=0\}}(t)\ dh^{*1}(t)$$

Here, h^{*1} is an arbitrary function, where the optimal solution is unique if and only if $G_X = 0$ is a Lebesgue null set. The minimum risk-adjusted liability is:

$$\inf_{f\in F} L_f(X) = \int_0^\infty \min\{r(S_X(t)),\ \delta g(S_X(t)) + (1-\delta)S_X(t)\}\ dt$$

5.2 Budget-constrained Problem

We are interested in the case where $\int_{G_X\le 0} r(S_X(t))dt > \pi$ when the insurance company is under a budget constraint and must be aware that the premium cannot exceed the budget. We need to find the most efficient method to cede the excess losses in the set $G_X \le 0$.

To accomplish this, we define the benefit-to-cost ratio $H_X : [0, \operatorname{ess\,sup}(X)) \to \mathbb{R}^+$ as follows:

$$H_X(t) = \frac{\delta g(S_X(t)) + (1-\delta)S_X(t)}{r(S_X(t))}$$

Notice that $H_X(t) \ge 1$ if and only if $G_X \le 0$.

Theorem 5.3. *Let:*

$$c^* = \inf\left\{c > 1 \,\middle|\, \int_{H_X\ge c} r(S_X(t))\ dt \le \pi\right\}$$

1. *If $\int_{G_X\le 0} r(S_X(t))\ dt \le \pi$, the optimal solution is the same as Theorem 5.2.*
2. *If $\int_{G_X<0} r(S_X(t))\ dt \le \pi < \int_{G_X\le 0} r(S_X(t))\ dt \le \pi$, then the optimal solution is of the form*

$$f^{*2}(x) = \int_0^x \mathbb{1}_{\{G_X<0\}}(t)\ dt + \int_0^x \mathbb{1}_{\{G_X=0\}}(t)\ dh^{*2}(t)$$

where h^{*2} is any function in $\mathcal{F}$ such that

$$\int_{G_X<0} r(S_X(t))\ dt + \int_{\{G_X=0\}} r(S_X(t))\ dh^{*2}(t) \le \pi$$

3. *If* $\int_{G_X<0} r(S_X(t))\ dt > \pi$ *and* $\int_{H_X>c^*} r(S_X(t))\ dt = \pi$*, then the unique optimal solution is of the form*

$$f^{*3}(x) = \int_0^x \mathbb{1}_{\{H_X \geq c^*\}}(t)\ dt$$

4. *If* $\int_{G_X<0} r(S_X(t))dt > \pi$ *and* $\int_{H_X>c^*} r(S_X(t))dt \leq \pi < \int_{H_X>c^*} r(S_X(t))\ dt$*, then every optimal solution is of the form*

$$f^{*4}(x) = \int_0^x \mathbb{1}_{\{H_X>c^*\}}(t)\ dt + \int_0^x \mathbb{1}_{\{H_X=c^*\}}(t)\ dh^{*4}(t)$$

where h^{*4} *is any function in* $\mathcal{F}$ *such that*

$$\int_{\{H_X>c^*\}} r(S_X(t))\ dt + \int_{\{H_X>c^*\}} r(S_X(t))\ dh^{*4}(t) = \pi$$

5.3 Cost-Benefit Numerical Analysis

Following the theme of the previous numerical studies, we use the exponential distribution to model losses. As such, we have:

$$F(x) = 1 - \lambda e^{-\lambda x}$$
$$S(x) = \lambda e^{-\lambda x}$$

Following the guidelines set by Theorem 5.2, we have the optimal function in the form:

$$f^{*1}(x) = \int_0^x 1_{\{G_X<0\}}(t)\ dt + \int_0^x 1_{\{G_X=0\}}(t)\ dh^{*1}(t)$$

where the function $G_X(t)$ is given as:

$$G_X(t) = r(S_X(t)) - [\delta g(S_X(t)) + (1-\delta)S_X(t)]$$

For our numerical example, we set the functions $r = (1+\theta)x$, $g = \sqrt{x}$, and $h^{*1} = \frac{x}{2}$. We begin by substituting the values into the function $G_X(t)$:

$$\begin{aligned} G_X(t) &= r(S_X(t)) - [\delta g(S_X(t)) + (1-\delta)S_X(t)] \\ &= (1+\theta)S_X(t) - [\delta\sqrt{(S_X(t))} + (1-\delta)S_X(t)] \\ &= (1+\theta)\lambda e^{-\lambda t} - [\delta(\sqrt{\lambda e^{-\lambda t}}) + (1-\delta)\lambda e^{-\lambda t}] \\ &= (1+\theta)\lambda e^{-\lambda t} - \delta(\sqrt{\lambda e^{-\lambda t}}) + (1-\delta)\lambda e^{-\lambda t}] \\ &= (\theta+\delta)\lambda e^{-\lambda t} - \ \delta\sqrt{\lambda e^{-\lambda t}} \end{aligned}$$

To solve for the condition where $G_X(t) = 0$, we set the above criterion equal to 0. We get:

$$\sqrt{\lambda e^{-\lambda t}} = \frac{\delta}{\theta+\delta}$$

When we solve for t:

$$t = -\frac{1}{\lambda}\ln\left(\frac{(\frac{\delta}{\theta+\delta})^2}{\lambda}\right)$$

To solve for the condition where $G_X(t) < 0$, we get:

$$t > -\frac{1}{\lambda}\ln\left(\frac{(\frac{\delta}{\theta+\delta})^2}{\lambda}\right)$$

We can now insert values for δ, θ, and λ.

δ	θ	λ
0.05	.5	.01

$$-\frac{1}{\lambda}\ln\left(\frac{(\frac{\delta}{\theta+\delta})^2}{\lambda}\right) = 19.062036$$

With the arbitrary function set at $h^{*1} = \frac{x}{2}$, and $h^{*1'} = \frac{1}{2}$, we get:

$$f^{*1}(x) = \int_{19.062}^{19.062} dt + \int_{19.062}^{x} dh^{*1}(t) = \int_{19.062}^{x} h^{*1'}(t)\ d(t)$$

Hence:

$$f^*(x) = \frac{1}{2}(x - 19.062)_+$$

This optimal solution is in the form $f^*(x) = c(x-d)_+$, which is a change loss reinsurance contract.

6 Conclusion

To recap, we looked at various risk measures and objective functions in determining the optimal ceded loss functions. We conducted multiple numerical analysis to discover how certain criteria affect the optimal solution. What we realized is that for different circumstances, there are different optimal contracts. Future research may continue to look at additional distortion risk measures to see how we may derive optimal solutions, as well as conduct studies to visually represent our findings.

References

[1] Artzner, P, Delbaen, F, Eber, J, Heath, D. "Coherent Risk Measures." *Mathematical Finance.* Volume 9, 1999.

[2] Cai, J, Tan, K, Weng, C, Zhang, Y. "Optimal reinsurance under VaR and CTE risk measures." *Insurance: Mathematics and Economics.* Volume 43, 2008.

[3] Cheung, K. “Optimal reinsurance revisited – a geometric approach.” *ASTIN Bulletin.* Volume 40, 2010.

[4] Cheung, K, Lo, A. "Characterizations of optimal reinsurance treaties: a cost-benefit approach." *Scandinavian Actuarial Journal.* Volume 28, 2015.

[5] Denuit, M, Dhaene, J, Goovaerts, M, Kaas, R. "Modeling and Measuring Risks." *Actuarial Theory for Dependent Risks: Measures, Orders and Models.* Wiley, 2005, pp. 3–102.

[6] Föllmer, H, Schied, A. "Convex measures of risk and trading constraints." *Finance and Stochastics.* Volume 6, 2002.