# BOUNDED COHOMOLOGY, AMENABILITY AND HYPERBOLICITY.

MATH 595, the first half of Spring 2018.

Igor Mineyev, 3:00 pm MWF, 347 Altgeld Hall.

First we will define *bounded cohomology* of groups and topological spaces, then exhibit its relations to analytic, geometric, and group theoretic concepts, notably *amenable groups* and *hyperbolic groups*. The beauty of the subject comes from the fact that it connects several areas of mathematics. Introduced by B.E. Johnson in the 70's under the name "cohomology of Banach algebras" (a purely analytic notion), bounded cohomology later attracted topologists/geometers, especially after a seminal paper "Volume and bounded cohomology" by Gromov. This relates it to the notion of *simplicial volume* of a manifold. In the case of a hyperbolic manifold, Gromov and Thurston showed that the simplicial volume (think of the surface of a pretzel) is proportional to the usual volume. This gives an alternative definition of volume that can be defined for *any* topological space. I will also present a characterization of hyperbolic groups in terms of bounded cohomology. Just as $\ell^\infty$ is dual to $\ell^1$, bounded cohomology is dual to *summable homology*, which is defined in a dual fashion.

An orange can be cut into finitely many rigid pieces and the parts can be rearranged so that the result is TWO oranges, each exactly the same as the original one. This is a mathematically correct statement if we assume the axiom of choice, which we all do. This statement is known as the Banach-Tarski paradox. A surprising ingredient in its proof is the use of group actions, specifically, actions of free groups. Why can such a strange thing happen to oranges? Because oranges, and most free groups, lack *amenability*. *Amenable groups* admit many equivalent definitions: combinatorial, measure-theoretic, in terms of $C^*$-algebras, etc. They are also characterized by *not* having the above Banach-Tarski paradox. Already in his original paper, Johnson gave a characterization of amenable groups via bounded cohomology. We will discuss this analytic notion, talk about bounded cohomology of free groups and other groups. One notoriously difficult open problem asks: Is the Thompson group amenable? If time allows, we will also mention results of Linnell and Morris that show that for amenable groups, left-orderability is equivalent to local indicability, and also to certain geometric conditions on the group.

There is no required prerequisite and no required textbook for the course.