Math 595 Calculus on Meshes

Spring 2018 (Half-semester: Second 8 Weeks)

Instructor: Anil Hirani, hirani@illinois.edu

Course Description: Introduction to finite element discretization of exterior calculus on piecewise linear manifold simplicial complexes (meshes). Approximate syllabus:

1. Finite element method (3 lectures)
   1.1 Introduction to Galerkin methods and finite elements (1 lecture)
   1.2 Consistency, convergence and stability of numerical methods (1 lecture)
   1.3 Computational examples (1 lecture)
2. Hilbert complexes (6 lectures)
   2.1 Basic definitions and Poincaré inequality (1 lecture)
   2.2 Hodge Laplacian and Hodge decomposition on Riemannian manifolds (1 lecture)
   2.3 Hodge decomposition in vector spaces and in Hilbert complexes (1 lecture)
   2.4 Abstract Hodge Laplacian and mixed formulation (1 lectures)
   2.5 Well-posedness of the mixed formulation (2 lectures)
3. Abstract approximation of Hilbert complexes (3 lectures)
   3.1 Bounded cochain projections and harmonic gap (1 lecture)
   3.2 Stability and convergence of the mixed method (2 lectures)
4. de Rham complex as a Hilbert complex (2 lectures)
   4.1 Square integrable forms and $H\Lambda^k$ spaces (1 lectures)
   4.2 Hodge Laplacian problems in 1, 2 and 3 dimensions (1 lecture)
5. Approximations of the de Rham complex (7 lectures)
   5.1 Polynomial differential forms ($\mathscr{P}_r^-\Lambda^k$ and $\mathscr{P}_r\Lambda^k$) and the Koszul complex (2 lectures)
   5.2 Lowest order Whitney forms ($\mathscr{P}_1^-\Lambda^k$ spaces) and their properties (2 lectures)
   5.3 Degrees of freedom and finite element differential forms (2 lectures)
   5.4 Constructing $\mathscr{P}_r^-\Lambda^k$ from barycentric coordinates (1 lecture)

"Any young (or not so young) mathematician who spends the time to master this paper will have tools that will be useful for his or her entire career." – AMS Mathematical Reviews, on the 2006 paper below.

Readings: Notes provided by instructor and two papers by Arnold, Falk and Winther: (i) *Finite element exterior calculus: from Hodge theory to numerical stability*, Bulletin of the AMS, 2010; and (ii) *Finite element exterior calculus, homological techniques, and applications*, Acta Numerica, 2006.

Grading: Homeworks and a term paper.

Prerequisites: This course is designed for mathematics graduate students with knowledge of calculus on manifolds (Math 481 or Math 518 or equivalent) and who have some basic knowledge of functional analysis at the level of definitions (operator norms, bounded and unbounded operators). Some basic definitions and examples needed from elementary algebraic topology (chains, cochains, boundary, coboundary, homology and cohomology) will be either covered or notes provided, depending on the background of the students. No prior knowledge of partial differential equations will be assumed or needed.