MATH 530 - Comprehensive Examination, August 2019

Instructions: Do any four of the following five problems. Indicate which problem you have omitted.

Here is a theorem that you may wish to quote in your solutions:

Minkowski's bound: Let $K/\mathbb{Q}$ be a finite extension of the rational numbers with degree n and let O_K be the set of algebraic integers in K and Δ the discriminant of O_K over the integers. Assume K has r embeddings into the real numbers and $2s$ embeddings into the complex numbers. Then every class of fractional ideals contains an ideal I in O_K that satisfies

$$|N_{K/\mathbb{Q}}(I)| \leq \frac{n!}{n^n} \left(\frac{4}{\pi}\right)^s |\Delta|^{1/2}.$$

1. Let $K = \mathbb{Q}(\sqrt{-51})$.

a) Give the ring of integers R of K.

b) Give the decompsition of the ideals $2R$ and $3R$ as products of prime ideals in R.

c) Determine the structure of the ideal class group of R.

2. Let $K = \mathbb{Q}(\sqrt{3}, \sqrt{7}, \sqrt{11})$ and let R be the ring of integers of K.

a) For a prime P of R that divides the rational prime p define the decomposition field and the inertia field.

b) Find the decomposition field and the inertia field associated to a prime $P|5$.

c) Find the decomposition field and the inertia field associated to a prime $P|3$.

3. Let K be a number field with ring of integers R.

a) State Dirichlet's Unit Theorem for the structure of the unit group E_K of R.

b) In b), c), d) let $K = \mathbb{Q}(\sqrt{-3}, \sqrt{-5})$.
Determine the minimal m such that $K \subset \mathbb{Q}(\omega_m)$, for $\omega = e^{2\pi i/m}$.

c) Describe the structure of E_K as an abstract group.

d) Explicitly find a subgroup of finite index for E_K.

4. Prove that for a number field K with ring of integers R, every nonzero prime ideal of R is a maximal ideal.

(Hint: Show that for every nonzero ideal I there exists nonzero $m \in \mathbb{Z}$ such that $mR \subset I \subset R$ and use that R is a free $\mathbb{Z}$-module of finite rank.)

5. Let f be the polynomial $x^4 + 2x^2 + 2 \in \mathbb{Q}[x]$.

a) Show that f is irreducible over $\mathbb{Q}$.

b) Find the degree of each irreducible factor of f in the polynomial ring $\mathbb{Q}_5[X]$. Here $\mathbb{Q}_5$ denotes the complete field of $5-$adic numbers.